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Meeting the load on a large scale power system network involves the scheduling and control of multiple generating units with the primary goal of minimizing the total cost of generation. Many constraints and factors must be taken into account and problems of scheduling involve times scales from several years for future construction planning, to a year for maintenance scheduling, to daily or weekly scheduling of the units which are available for meeting the current demand. A totally integrated scheduling methodology taking into account all such factors involves computational requirements far beyond the practical

20. ABSTRACT (continued)

limitations of computing technology. Consequently, each aspect of the problem has been previously investigated with the relationship to the other time scales being only heuristicly and intuitively involved.

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SCHEDULING AND COORDINATION OF MULTIPLE DYNAMIC SYSTEMS

by

Bruce Krogh

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1. INTRODUCTION

1.1. Motivating Problem - Unit Commitment in Power Systems

Meeting the load on a large scale power system network involves the scheduling and control of multiple generating units with the primary goal of minimizing the total cost of generation. Many constraints and factors must be taken into account and problems of scheduling involve time scales from several years for future construction planning, to a year for maintenance scheduling, to daily or weekly scheduling of the units which are available for meeting the current demand. A totally integrated scheduling methodology taking into account all such factors involves computational requirements far beyond the practical limitations of computing technology. Consequently, each aspect of the problem has been previously investigated with the relationship to other time scales being only heuristicly and intuitively involved.

One of these problems has become known as the unit commitment problem. This deals with the scheduling on a daily or weekly basis of the generating units which are to supply the power to meet the predicted demand. Since various units operate at different efficiencies, the cost of meeting the demand can be affected by the selection of the generating units to be on line at any given time. Obviously, the availability of units is affected by maintenance scheduling and forced outages. Assuming such schedules are known, the commitment problem involves the selection of the optimal generation schedule from an enormous number of possibilities. This problem itself is much too large to be solved completely for a practical system. Therefore, suboptimal schemes have been suggested and implemented for current applications [1-10].

Currently, unit commitment is done using interactive computer programs which allow the system operator to employ the computer to enhance his own intuitive knowledge and experience concerning reasonable generation schedules [7]. A relatively small amount of theoretical investigation has been reported concerning many important aspects of the problem. The suboptimal schemes which have been researched and implemented certainly use sophisticated optimization theory whenever possible, including integer programming [2,5,9,10] and dynamic programming [4]. However, the approach normally taken has been to cast the unit commitment problem in a form which allows the application of existing optimization theory. Only recently has the particular problem of unit commitment been used to motivate research which seeks new ideas to include important aspects of the problem's unique nature.

In two papers, Turgeon [11,12], suggests that the shut-down and start-up of thermal units may be modelled dynamically and that the control of the off-line units may be considered as individual optimization problems. He suggests that the systems which are on-line operate at a given state which determines the end conditions for the control problem defined for the off-line systems. In the following, these concepts are incorporated into the generic model for a general scheduling and coordination problem for multiple dynamic systems. The issues addressed are motivated by the unit commitment problem, but are of a strictly theoretical nature. The motivation for such an investigation is two-fold.

Firstly, the general problem formulation demands a fundamental investigation involving basic properties which are assumed or overlooked in more pragmatic approaches to the problem. Often these properties are intuitively obvious and hence not of necessary consequence when the goal of research is to produce a working, practical suboptimal scheduling scheme. As is born out in the following, such properties often require considerable care and effort to state and demonstrate rigorously. Secondly, the results presented are directed towards implementation as another step in the development of practical computational schemes to solve the unit commitment problem. Since all current methods produce suboptimal solutions and the global optimization problem remains impractically large, the results presented here enable one to take any given schedule and to improve upon it through a type of gradient search. The theoretical analysis of such an approach is interesting and useful in its own right.

In the remainder of section 1, the generic optimization problem to be considered throughout the report is first described in words and then formulated mathematically. It will be clear that the unit commitment problem for power systems discussed above directly motivates and can be formulated within the general model given. However, since the primary interest of the research was of a theoretical nature the unit commitment problem and the application of the results presented to that problem will not be mentioned until the concluding section of the report. In that section, the application of the theoretical work and some practical aspects of the thermal unit modeling will be discussed.

1.2. Generic Problem Description

Large scale systems often consist of several subsystems which must be operated in a coordinated fashion so that the aggregate performance of the subsystems meets a given objective. The particular structure of the large scale systems to be considered in the following involves several essentially autonomous dynamic subsystems for which the local controls are selected in response to a policy dictated by an overall coordinator. Each subsystem may produce an output over a certain range of output levels while maintaining a particular operating steady-state. The coordinator selects the subsystem output levels so that the sum of the outputs from the participating subsystems equals a given demand to be met by the large scale systems (Fig. 1).

Since the subsystems operate at a steady-state while meeting the output level dictated by the coordinator, the coordination of the participating systems is determined so as to minimize the total steady-state operating costs of all subsystems involved. Hence, even if the overall demand varies with time, the optimal output level for each participating unit may be selected by solving a static optimization problem for each point in time.

The coordinator is allowed at each point in time the option of not including some of the subsystems at all in the steady-state output coordination. If for an interval of time a particular subsystem is released from participating in the aggregate output of the overall system, it becomes autonomous and need not maintain the operating steady-state until the coordinator once again demands an output from it. Hence, the local control is selected so that the subsystem is back in the operating steady-state when needed. It is assumed there is cost incurred in operating the subsystem during such an "off-line" interval of time and the local control is determined so as to minimize this cost.

The coordination policy involves two levels, the second being determined in light of the first. The coordinator must select the subsystems

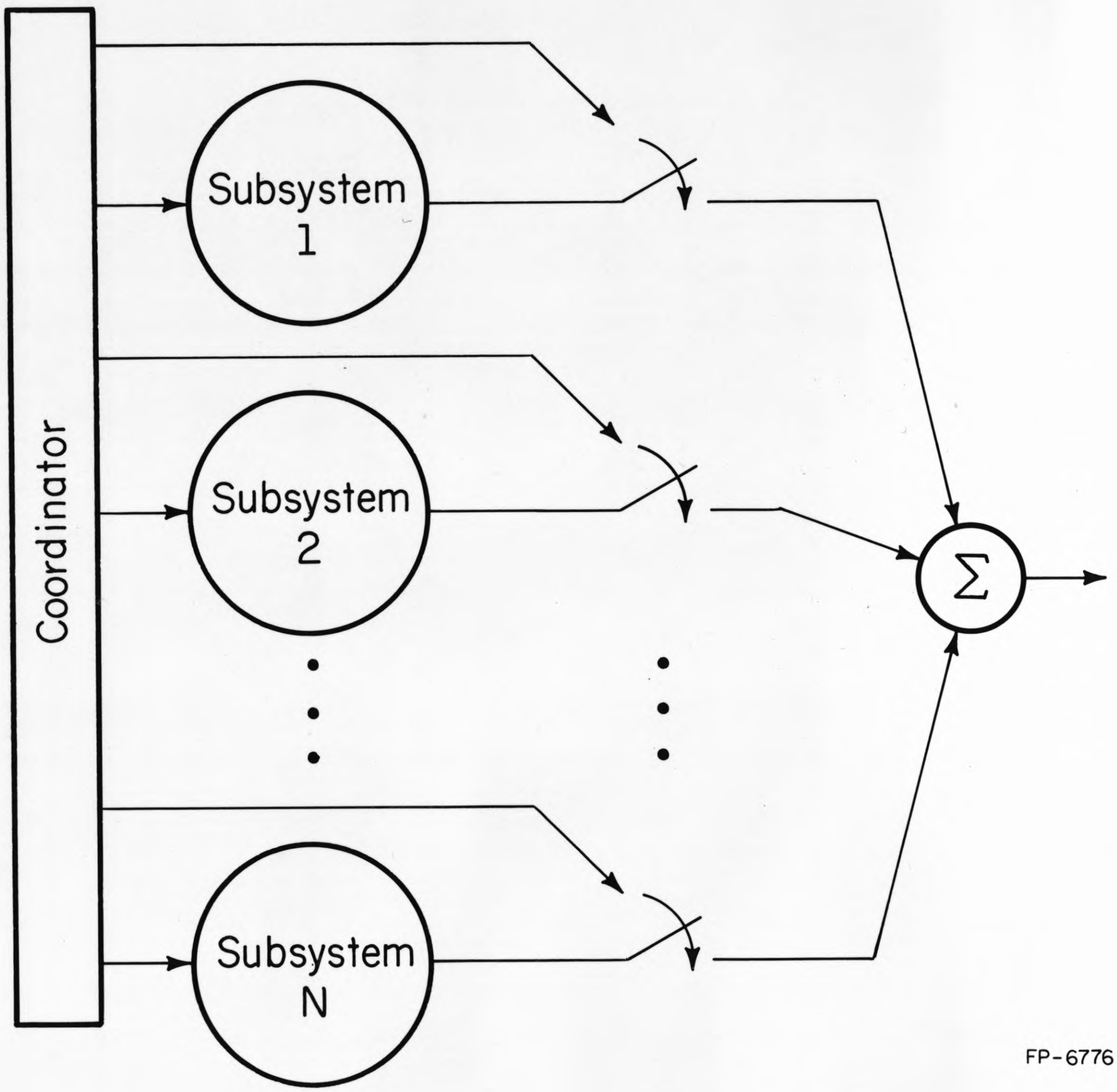


Figure 1. Large Scale System Structure.

which will participate in the output production for each point in time. Then, each local control is determined to either produce an output which minimizes the overall steady-state production cost or, during, the intervals of time when the subsystem is off-line, the local control is determined by a dynamic optimization. The coordinator must select the participation policy so as to minimize the total cost incurred by all subsystems during both the off-line and production or "on-line" time intervals.

1.3. Mathematical Model

Suppose the system is to operate during the time interval $[0,T]$ and the demand to be met by the sum of the subsystem outputs is given by the continuous function $D:[0,T] \to R^1$. The dynamics of the N subsystems are described by the equations

$$\dot{x}^i = f^i(x^i, u^i, d_i \cdot v_i) \qquad i=1,\ldots,N \tag{1}$$

where x^i and u^i are vector functions of time representing the state of the ith subsystem and its local control, respectively, with $x^i(t) \in R^{n_i}$ and $u^i(t) \in R^{m_i}$. The scalar function $d_i(t)$ is the output demanded from the ith subsystem when it is on-line and the coordinator must select $\underline{d}(t) = \begin{bmatrix} d_1(t) \\ \vdots \\ d_N(t) \end{bmatrix}$ to satisfy the constraints

$$\underline{d}_i \le d_i(t) \le \overline{d}_i, \qquad i=1,\ldots,N, \qquad t \in [0,T]. \tag{2}$$

The scalar function $v_i : [0,T] \to \{0,1\}$ is selected by the coordinator to indicate when the ith subsystem is on-line or off-line according to

$$v_i(t) = \begin{cases} 0 & \to \text{ ith subsystem off-line} \\ 1 & \to \text{ ith subsystem on-line.} \end{cases} \tag{3}$$

Hence, when the ith subsystem is off-line the output demand is $d_i(t)\cdot v_i(t) = 0$.

Each subsystem can meet the demand $d_i(t)$ when it is on-line and in an operating steady-state denoted by $\bar{x}^i \in R^{n_i}$, i=1,...,N. The control at each point in time must be in a given constraint set $U^i \subset R^{m_i}$, i=1,...,N. Hence, for all d_i satisfying (2) there is a $u^i \in U^i$ such that

$$0 = f^i(\bar{x}^i, u^i, d_i) \tag{4}$$

which, because of (1), implies steady-state operation. Although there may be many admissable u^i which satisfy (4) for a given d_i, it is assumed that for each demand d_i there is a predetermined, unique u^i which is selected as the local control to satisfy (4). I.e., there is a function $\Psi^i : R^1 \to R^{m_i}$ which determines the on-line control for the ith subsystem given the demand, d_i. Therefore,

$$f^i(\bar{x}^i, \Psi^i(d_i), d_i) = 0 \quad \forall \quad \underline{d}_i \le d_i \le \bar{d}_i. \tag{5}$$

Let the cost per unit time of operating the ith subsystem with control u^i while in state x^i be given by the scalar valued function, $L_i : R^{n_i+m_i} \to R^1$. Then for steady-state operation the cost per unit time is given by $L_i(\bar{x}^i, u^i)$, or the cost can be written directly as a function of the demand d_i, $\varphi_i(d_i)$, given by

$$\varphi_i(d_i) = L_i(\bar{x}^i, \Psi^i(d_i)). \tag{6}$$

Now, for a given time $t \in [0,T]$, suppose the coordinator has determined the subsystems which are to be on line by selecting the appropriate $\underline{v}(t) \in \{0,1\}^N$ where $\underline{v}(t) = [v_1(t),\ldots,v_N(t)]$ and the $v_i(t)$ are defined by (3).

Let $\mathcal{P}[M]$ denote the power set of an arbitrary set M, i.e. $\mathcal{P}[M]$ is the collection of all subsets of M. Define $\mathcal{S} : \{0,1\}^N \rightarrow \mathcal{P}[\{1,\ldots,N\}]$ as

$$\mathcal{S}(\underline{v}) = \{i \mid v_i = 1\} \quad \text{where } \underline{v} = [v_1,\ldots,v_N]' \in \{0,1\}^N. \tag{7}$$

For a given demand $D(t) = \hat{D}$, the vector $\underline{v}(t) = \hat{\underline{v}}$ will be said to be feasible for $\hat{D}$ if

$$\sum_{i \in \mathcal{S}(\hat{\underline{v}})} \underline{d}_i \leq \hat{D} \leq \sum_{i \in \mathcal{S}(\hat{\underline{v}})} \overline{d}_i. \tag{8}$$

Clearly $\underline{v}(t)$ must be feasible for $D(t)$ for all $t \in [0,T]$ and this is a constraint imposed upon the selection of $\underline{v}(t)$.

Given a feasible $\underline{v}(t)$ for $D(t)$ the demands for the on-line units are chosen to minimize the sum of the costs for those units. Let $C : R^{N+1} \rightarrow R^1$ be the minimal cost per unit time for meeting $D(t)$ with the subsystems $\mathcal{S}[\underline{v}(t)]$. I.e.

$$C[D(t),\underline{v}(t)] = \min_{\underline{d}_i \leq d_i \leq \overline{d}_i} \sum_{i \in \mathcal{S}[\underline{v}(t)]} \varphi_i(d_i) \tag{9}$$

subject to the constraint

$$\sum_{i \in \mathcal{S}[\underline{v}(t)]} d_i = D(t) \tag{10}$$

Consider now the problem faced by the controller of the ith subsystem when $v_i(t) = 0$ on an interval $t \in (t_1,t_2)$ and $v_i(t)$ switches from 1 and to 1 at times t_1 and t_2, respectively. The control u^i on the interval (t_1,t_2) must minimize the cost

$$J_i(t_2-t_1,u^2) = \int_{t_1}^{t_2} L_i(x^i,u^i)dt \tag{11}$$

subject to the two point boundary value (tpbv) problem

$$\dot{x}^i = f^i(x^i,u^i,0)$$

with

$$x^i(t_1) = \bar{x}^i \quad \text{and} \quad x^i(t_2) = \bar{x}^i.$$

Let the minimal value of $J_i(t_2-t_1,u^i)$ be denoted by $J_i^*(t_2-t_1)$ if it exists. I.e.

$$J_i^*(t_2-t_1) = \min_{u^i(t)\in U^i} J_i(t_2-t_1,u^i). \tag{12}$$

Since all constraints and the cost per unit time, $L_i(x^i,u^i)$ are independent of time, it suffices to consider $J^*(\cdot)$ as a function of the time interval length, t_2-t_1.

To write the total cost of selecting a particular policy $\underline{v}(t)$ for $t\in[0,T]$, assuming $\underline{v}(t)$ is feasible for each $D(t)$, let $\{\tau_k^i\}_{k=1}^{N_i}$ be the times for which $v_i(t)$ switches from 0 to 1, and let $\{\lambda_k^i\}_{k=1}^{N_i}$ be the times $v_i(t)$ switches from 1 to 0. Assume

$$0 \le \lambda_1^i < \tau_1^i < \lambda_2^i < \tau_2^i < \ldots < \lambda_{N_i}^i < \tau_{N_i}^i \le T \tag{13}$$

which is always possible for any $v_i(t)$ (since, if it is not true, (13) may be made to hold for another $\hat{\underline{v}}(t)$ for which $\hat{v}_i(t) = v_i(t)$ almost everywhere (a.e.)). Then the total cost of the off-line operation of the ith subsystem, $\mathcal{J}_i[v_i(t)]$, which is actually a functional of $v_i(t)$ for $0\le t\le T$, is given by

$$\mathcal{J}_i[v_i(t)] = \sum_{k=1}^{N_i} J_i^*(\tau_k^i - \lambda_k^i). \tag{14}$$

So, letting $\mathcal{C}[\underline{v}(t)]$ denote the total cost for both the off-line and on-line systems over [0,T] for policy $\underline{v}(t)$, we have

$$\mathcal{C}[\underline{v}(t)] = \int_0^T C[D(t),\underline{v}(t)]dt + \sum_{i=1}^{N} \mathcal{J}_i[v_i(t)]. \tag{15}$$

The coordinator wishes to select $\underline{v}(t)$ to minimize $\mathcal{C}[\underline{v}(t)]$ of (15). Note that the definitions of $C[D(t),\underline{v}(t)]$ and $\mathcal{J}_i[v_i(t)]$ actually involve the costs incurred by the subsystems and the local control policies, $u^i(t)$. However, once $\underline{v}(t)$ is selected, $u^i(t)$ is determined directly from the steady-state and off-line optimization problems defined above. Hence, the coordinator selects the policy $\underline{v}(t)$ to minimize $\mathcal{C}[\underline{v}(t)]$ subject to the subsequent optimization problems which determine the local controls, $u^i(t)$. This is a leader-follower differential game which happens to be of a special sort defined and discussed in the following section.

2. RELATIONSHIPS BETWEEN LEADER-FOLLOWER AND TEAM GAMES

2.1. General Results

It is well-known from the theory of leader-follower games that, in general, the minimal cost obtainable by the leader is not necessarily the absolute minimum cost which would be obtainable were the leader and followers to cooperate in selecting policies with the sole objective of minimizing the leader's cost. This latter situation where all participants in a game cooperate to minimize a single cost will be referred to as an optimal team policy. The following discussion leads to a general condition for which an optimal leader-follower policy actually gives a solution which is equivalent to an optimal team policy with respect to the leader's cost.

Consider the following two problem definitions, Pr 1 and Pr 2.

Pr 1. A game with N+1 participants is defined as follows. One player, the leader, selects a policy, v, from a decision space V and the remaining N players, the followers select their policies, u^i, from decision spaces $U^i(v)$, i=1,...,N. As indicated by the notation, the decision space for the ith follower depends upon the particular policy selected by the leader, but is independent of the policies of the other N-1 followers. Each follower chooses its policy once the leader's policy is declared to minimize, if possible, the cost $J_i(v,u^i)$ which, as indicated, is a function of the leader's policy and the particular follower's policy, but is independent of the other N-1 followers' policies. Define the optimal followers' costs given v, $J_i^*(v)$, as

$$J_i^*(v) = \inf_{u^i \in U^i(v)} J_i(v,u^i) \quad i = 1,\dots,N. \tag{16}$$

It is assumed the infinum exists in (16) and subsequent definitions in the sequel.

Define $T_i : v \to \mathcal{P}\{u^i(v)\}$ for i=1,...,N as

$$T_i(v) = \{u^i \in U^i(v) \mid J_i(v,u^i) = J_i^*(v)\}. \tag{17}$$

Note that it is possible for $T_i(v) = \phi$ for some $i \in \{1,\dots,N\}$ and $v \in V$.

The leader wishes to minimize, if possible, a cost, $J_o(v,\underline{u})$, where $\underline{u} = [u^1,\dots,u^N] \in \underline{U}(v)$ and $\underline{U}(v) = U_1(v) \times \dots \times U_N(v)$. Define the set $\hat{V} \subset U$ as

$$\hat{V} = \{v \in V \mid T_i(v) \neq \phi \quad \forall\ i=1,\dots,N\}. \tag{18}$$

The leader will only consider policies in $\hat{V}$, i.e. policies for which all followers can achieve their optimal costs. If $\hat{V} = \phi$, the problem will be considered ill-posed and such cases will not be included in the subsequent discussion.

The optimal value of $J_o(v,\underline{u})$ for the leader-follower game will be denoted by J^*_{L-F} and is defined as

$$J^*_{L-F} = \inf_{\substack{v \in \hat{V} \\ \underline{u} \in \underline{T}(v)}} J_o(v,\underline{u}) \tag{19}$$

where $\underline{\mathbf{T}} : V \rightarrow \mathcal{P}\{U^1(v)\} \times \mathcal{P}\{U^2(v)\} \times \ldots \times \mathcal{P}\{U^N(v)\}$ is defined by

$$\underline{T}(v) = T_1(v) \times T_2(v) \times \ldots \times T_N(v). \tag{20}$$

A set of policies, $(v,\underline{u})$, for the N+1 players will be called a solution to Pr 1 if $v \in \hat{V}$, $\underline{u} \in \underline{T}(v)$, and $J_o(v,\underline{u}) = J^*_{L-F}$. The solution set of Pr 1, Σ_1, is then given by

$$\Sigma_1 = \{(v,\underline{u}) \in V \times \underline{U}(v) \mid v \in \hat{V},\ \underline{u} \in \underline{T}(v),\ J_o(v,\underline{u}) = J^*_{L-F}\}. \tag{21}$$

Pr 2. A game with N+1 participants is defined with policies v and $\underline{u}$ and policy spaces V and $\underline{U}(v)$ as defined for Pr 1. An optimal team policy is sought to minimize $J_o(v,\underline{u})$ defined in Pr 1. The optimal value for this team game is denoted by J^*_T and is given by

$$J^*_T = \inf_{\substack{v \in V \\ \underline{u} \in \underline{U}(v)}} J_o(v,\underline{u}). \tag{22}$$

A set of policies, $(v,\underline{u})$, for the N+1 players will be called a solution to Pr 2 if $v \in V$, $\underline{u} \in \underline{U}(v)$ and $J_o(v,\underline{u}) = J^*_T$. The solution set of Pr 2, Σ_2, is then given by

$$\Sigma_2 = \{(v,\underline{u}) \in V \times \underline{U}(v) \mid J_o(v,\underline{u}) = J^*_T\}. \tag{23}$$

The following two lemmas are immediately evident from the above definitions.

Lemma 1. For J^*_{L-F} and J^*_T defined by (19) and (22), respectively,

$$J^*_T \le J^*_{L-F}. \tag{24}$$

Proof. (24) follows from the fact that the infimum in (22) is taken over a larger class of $(v,\underline{u})$ than that of (19).

Lemma 2. If $J^*_T = J^*_{L-F}$ then

$$\Sigma_1 \subset \Sigma_2, \tag{25}$$

where Σ_1 and Σ_2 are defined by (21) and (23), respectively.

Proof. If $(v,\underline{u}) \in \Sigma_1$, then $v \in \hat{V} \subset V$ and $\underline{u} \in \underline{T}(v) \subset \underline{U}(v)$, and $J_o(v,\underline{u}) = J^*_{L-F} = J^*_T$, by hypothesis, therefore $(v,\underline{u}) \in \Sigma_2$, proving (25).

Now, consider the class of problems of the type Pr 1 for which the following property holds. Define the mappings $\Gamma_j : V \to \mathcal{P}\{U^1(v)\} \times \ldots \times \mathcal{P}\{U^N(v)\}$, $j=1,2$ as

$$\Gamma_1(v) = \{\underline{u} \in U(v) \mid u_i \in T_i(v) \;\; \forall\; i=1,\ldots,N\} \tag{26}$$

and

$$\Gamma_2(v) = \{\underline{u} \in U(v) \mid J_o(v,\underline{u}) = \inf_{\underline{u} \in \underline{U}(v)} J_o(v,\underline{u})\}. \tag{27}$$

Property π: A problem of the type Pr 1 will be said to have property π if

$$\Gamma_1(v) = \Gamma_2(v) \quad \forall \;\; v \in V. \tag{28}$$

In words, property π says that for every policy, $v \in V$, of the leader, a set $\underline{u} \in \underline{U}(v)$ of followers' policies minimizes their individual costs if and only if the leader's minimal possible cost for that v is also achieved by $\underline{u}$.

Theorem 1. If a problem of type Pr 1 has property π and either

$$\Sigma_2 \neq \phi \tag{29}$$

or

$$\hat{V} = V, \tag{30}$$

then

$$J_T^* = J_{L-F}^* \tag{31}$$

and

$$\Sigma_1 = \Sigma_2. \tag{32}$$

Proof. Suppose $(\tilde{v},\tilde{\underline{u}}) \in \Sigma_2$, then by (23) and (22)

$$J_T^* = J_o(\tilde{v},\tilde{\underline{u}}) = \inf_{\underline{u} \in U(\tilde{v})} J_o(\tilde{v},\underline{u}).$$

Therefore, by (27) and property π, $\tilde{\underline{u}} \in \Gamma_2(\tilde{v}) = \Gamma_1(\tilde{v})$ which implies $\tilde{v} \in \hat{V}$ and $\tilde{\underline{u}} \in \underline{T}(\tilde{v})$. Therefore, by the definition (19) of J_{L-F}^*,

$$J_{L-F}^* \leq J_o(\tilde{v},\tilde{\underline{u}}) = J_T^*.$$

But $J_T^* \leq J_{L-F}^*$ by lemma 1, hence (31) holds. Furthermore, $J_o(\tilde{v},\tilde{\underline{u}}) = J_{L-F}^*$, implying $(\tilde{v},\tilde{\underline{u}}) \in \Sigma_1$. Therefore, $\Sigma_2 \subset \Sigma_1$, but by (31) and lemma 2 $\Sigma_1 \subset \Sigma_2$, which establishes (32).

Now, assume $\hat{V} = V$. Note first that if (31) is established for this case, then lemma 2 implies (32). By lemma 1, $J_T^* \leq J_{L-F}^*$. Assume $J_T^* < J_{L-F}^*$. By the definition, (22) of J_T^* $\exists\, \hat{v} \in V$ and $\hat{\underline{u}} \in \underline{U}(\hat{v}) \ni J_T^* \leq J_o(\hat{v},\hat{\underline{u}}) < J_{L-F}^*$. Consider $\Gamma_1(\hat{v})$. Since $V = \hat{V}$, $\Gamma_1(\hat{v})$ is non-empty, hence $\exists$ $\hat{\underline{u}}' \in \Gamma_1(\hat{v})$. By property π, $\hat{\underline{u}}' \in \Gamma_2(\hat{v})$ which implies

$$J_o(\hat{v},\hat{\underline{u}}') \leq J_o(\hat{v},\hat{\underline{u}}) < J_{L-F}^*.$$

But $\hat{\underline{u}}' \in \underline{T}(\hat{v})$ and $\hat{v} \in \hat{V}$, so this contradicts the definition of J^*_{L-F}. Therefore, (31) and the theorem is proved.

2.2. Relationship to the Problem of Section 1

Theorem 1 establishes conditions for which the leader-follower game is virtually equivalent to a team game with respect to the leader's cost. It is easily seen that when the leader's cost consists of the sum of the individual followers' costs and the followers' problems are independent of one another, as is the case for the problem described in the previous section, then the problem is of type Pr 1 and has property π. Furthermore, for the problem models used in the sequel, it is easily shown that hypothesis (29) of theorem 1 as well as hypothesis (30) hold. Hence, for the problem being considered in this work, the leader in fact obtains the minimal possible cost even though the local controls are selected to solve peripheral optimization problems.

Although the result of theorem 1 may appear to be intuitively evident for a problem with the structure described in the previous section, the property π is of a very general nature and may often hold for leader-follower problems where the relationship between the optimal leader-follower policy and the optimal team policy is not immediately evident. It is often the case that the hypothesis for theorem 1 may be easily verified by applying known results in control and optimization theory to the given problem structure.

3. ANALYSIS OF THE COORDINATION OF THE ON-LINE SYSTEMS

3.1. Discussion and Primary Results

In this section the properties of the cost incurred by meeting the demand through steady-state optimal coordination of the on-line systems is considered. For a given feasible v with respect to a demand D, the minimal steady-state cost of meeting the demand, $C(D,\underline{v})$, is defined in (9) with the constraint (10). Since the optimization is for steady-state operation, the integral of $C[D(t),v(t)]$ appearing in the total cost function of (15) is minimized by minimizing the integrand for each $t \in [0,T]$.

The properties of $C[\cdot,\underline{v}]$ for a constant $\underline{v}$ are of interest for analysis of the overall optimization problem. Knowledge of the nature of this function allows one to assess what type of approaches should be used to solve the problem. In the following the major properties of $C[\cdot,\underline{v}]$ are given in a lemma. This result is discussed as it relates to a particular approach to minimizing the overall cost. Theorem 2, proved in this section, can be used to generate the gradient of the cost with respect to the switch times of $\underline{v}(t)$. This approach to the unit commitment problem will be discussed in Section 5.

The remaining primary result, theorem 3, describes the properties of the function $C^*(\cdot)$ which is defined below as the minimum cost to meet a demand D with any feasible $\underline{v}$. This function and its properties are of interest because it represents the optimization problem when the cost of operating the off-line systems is disregarded. Hence it provides a lower bound on the achievable cost. Furthermore, its properties are useful in guiding the search for reasonable solutions which employ integer or dynamic programming, as will be discussed in Section 5.

Following the presentation of the major results, the full proofs and intermediate results are presented in Section 3.2.

Lemma. (Lemma 6 in Section 3.2). If $\varphi_i(t)$, $i=1,\ldots,N$ of (6) are monotone increasing and continuous for each i, then $C(\cdot,\underline{v})$ is monotone increasing and continuous on any interval for which $\underline{v} \in \{0,1\}^N$ is feasible.

The interval of feasibility for $\underline{v}$ is discussed in Section 3.2.

Knowledge of the properties of the optimal cost for meeting the demand as a function of the demand will be useful in determining efficient algorithms for minimizing the overall cost. The following result is a consequence of the continuity of $C(\cdot,v)$.

Suppose for $D(t)$, $t\in[0,T]$, with range $[\underline{D},\overline{D}]$ (the range is defined in Section 3.2), there exists $\underline{v}_1$, $\underline{v}_2\in\{0,1\}^N$ such that both $\underline{v}_1$ and $\underline{v}_2$ are feasible for the entire range of demand. Define $\underline{v}(t,t_s)\in\{0,1\}^N$ for a given $t_s\in(0,T)$ and every $t\in[0,T]$ as

$$\underline{v}(t,t_s) = \begin{cases} \underline{v}_1 & 0 \leqslant t \leqslant t_s \\ \underline{v}_2 & t_s < t \leqslant T \end{cases}$$

Then $v(t,t_s)$ is feasible for $D(t)$ for all $t\in[0,T]$ and the cost of the steady-state coordination as a function of t_s, $\hat{J}(t_s)$, is given by

$$\hat{J}(t_s) = \int_o^T C[D(t),\underline{v}(t,t_s)]dt.$$

The derivative of $\hat{J}(t_s)$ exists and is given by the following.

<u>Theorem 2</u>. For $\hat{J}(t_s)$ and $\underline{v}(t,t_s)$ defined above, for any $\hat{t}\in(0,T)$,

$$\frac{\partial\hat{J}(t_s)}{\partial t_s}\Big|_{t_s=\hat{t}} = C[D(\hat{t}),\underline{v}_1] - C[D(\hat{t}),\underline{v}_2]. \tag{*}$$

<u>Proof</u>. For any $\hat{t}\in(0,T)$, and $0 < \Delta < T-\hat{t}$,

$$\hat{J}(\hat{t}+\Delta) - \hat{J}(\hat{t}) = \int_o^T\{C[D(t),\underline{v}(t,\hat{t}+\Delta)] - C[D(t),\underline{v}(t,\hat{t})]\}dt$$

$$= \int_t^{\hat{t}+\Delta}\{C[D(t),\underline{v}_1] - C[D(t),\underline{v}_2]\}dt = \{C[D(\tilde{t}),\underline{v}_1] - C[D(\tilde{t}),\underline{v}_2]\}\cdot\Delta$$

for some $\tilde{t}\in[\hat{t},\hat{t}+\Delta]$, where the last line follows from the continuity of $C(\cdot,\underline{v}_i)$, $i=1,2$, and the mean value theorem. Dividing by Δ and taking the limit as $\Delta \to 0$ gives (*) for the right derivative. The left derivative is proven to be equivalent to (*) in a similar manner. Q.E.D.

A generalization of Theroem 2, combined with some results of Section 4, can be used to define the gradient of the total cost (15) with respect to the switch times of an arbitrary v(t). This approach to the optimization problem is discussed in Section 5.

The other major result proved in Section 3.2 is given by the following theorem.

Theorem 3. If $C^*(\cdot)$ is discontinuous at $\tilde{D} \in [\underline{D}, \overline{D}]$ then the discontinuity is one of the following two types:

Type 1. $\exists\, S_1 \in \mathcal{P}_N$ with $\underline{v}_1 = \mathcal{S}^{-1}(S_1) \ni$

$\tilde{D} = \sum_{i \in S_1} \underline{d}_i$, $C^*(\tilde{D}) = C(\tilde{D}, \underline{v}_1)$ and

$C^*(\tilde{D}^-) > C^*(\tilde{D})$.

Type 2. $\exists\, S_2 \in \mathcal{P}_N$ with $\underline{v}_2 = \mathcal{S}^{-1}(S_2) \ni$

$\tilde{D} = \sum_{i \in S_2} \overline{d}_i$, $C^*(\tilde{D}) = C(\tilde{D}, \underline{v}_2)$ and

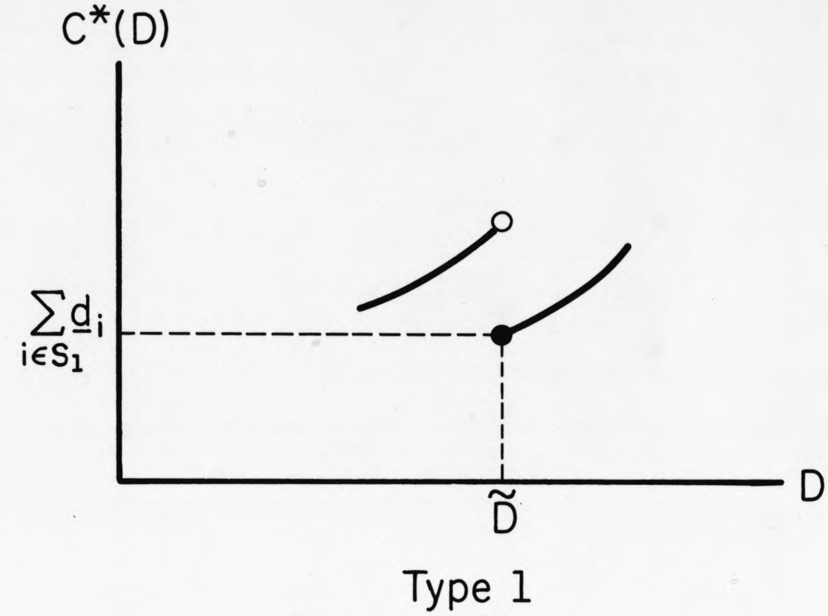
$C^*(\tilde{D}^+) > C^*(\tilde{D})$.

Furthermore, $C^*(\cdot)$ is monotone increasing on any interval in $[\underline{D}, \overline{D}]$ which contains no points of discontinuity of $C^*(\cdot)$.

See Figure 2 illustrating the two types of discontinuity for $C^*(\cdot)$. In words, theorem 3 states that at points of discontinuity $C^*(\cdot)$ assume the lower value of the right and left limits. Definitions of certain notation in Theorem 3 are given in Section 3.2.

3.2. Proofs of Primary Results

Before considering the properties of $C[D, \underline{v}]$, it is necessary to establish conditions for which there is at least one feasible $\underline{v} \in \{0,1\}^N$ for



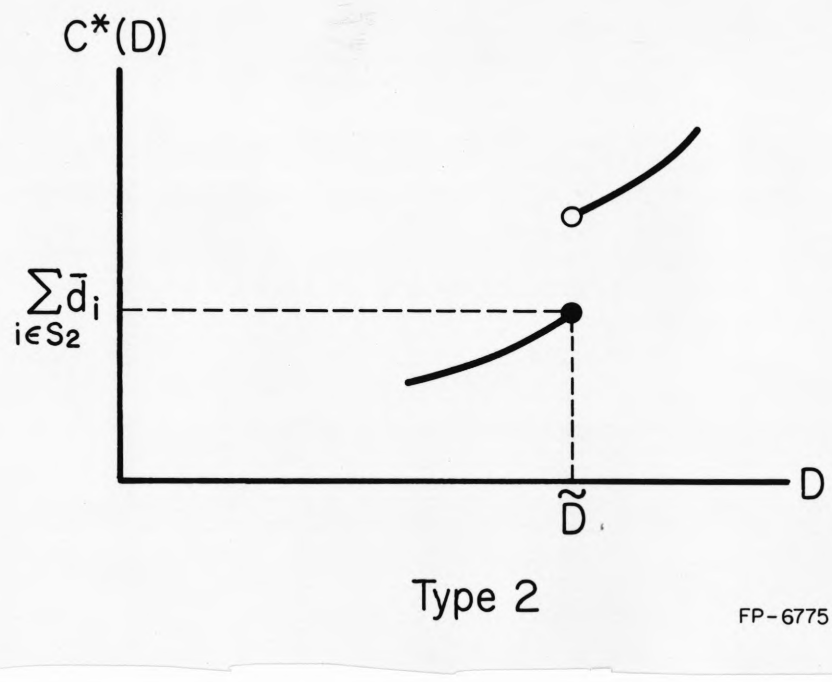


Figure 2. Types of Discontinuity of $C^*(D)$.

every value of $D(t)$, $t \in [0,T]$. Let $\underline{D} = \min_{t \in [0,T]} D(t)$ and $\overline{D} = \max_{t \in [0,T]} D(t)$, both of which exist since $D(t)$ is continuous. The range of demand is defined as the closed interval $[\underline{D},\overline{D}]$.

Definition. A set of N subsystems is said to have sufficient capacity with respect to (WRT) the range of demand $[\underline{D},\overline{D}]$ if $\forall D \in [\underline{D},\overline{D}]$ $\exists S \in \mathcal{P}_N$, where $\mathcal{P}_N = \mathcal{P}[\{1,\ldots,N\}]$, $\ni$

$$\sum_{i \in S} \underline{d}_i \leq D \leq \sum_{i \in S} \overline{d}_i. \tag{33}$$

An equivalent statement for the above definition would be to say there is a feasible $\underline{v} \in \{0,1\}^N$ $\forall$ $D \in [\underline{D},\overline{D}]$, which follows from (8). Both the definition of a feasible $\underline{v}$ and the definition above of sufficient capacity are based upon the simple fact stated in the following lemma.

Lemma 3. Given a set of subsystems, $S \in \mathcal{P}_N$, and a demand D, $\exists d_i \ni \underline{d}_i \leq d_i \leq \overline{d}_i$ for each $i \in S$ where $\sum_{i \in S} d_i = D$ if D and S satisfy (33).

Proof. Trivial.

Next we state a necessary and sufficient condition for a set of subsystems to have sufficient capacity with respect to a given range of demand.

Lemma 4. A set of N subsystems has sufficient capacity WRT the range of demand $[\underline{D},\overline{D}]$, if $\exists Q \in \mathcal{P}_N \ni$

$$\sum_{i \in Q} \underline{d}_i \leq \underline{D} \leq \sum_{i \in Q} \overline{d}_i \tag{34}$$

and $\forall S_1 \in \mathcal{P}_N \ni$

$$\underline{D} \leq \sum_{i \in S_1} \overline{d}_i < \overline{D} \tag{35}$$

$\exists S_2 \in \mathcal{P}_N$ which may depend upon $S_1 \ni$

$$\sum_{i \in S_2} \underline{d}_i \leq \sum_{i \in S_1} \overline{d}_i < \sum_{i \in S_2} \overline{d}_i.$$

Proof. Necessity. Clearly (34) holds by the definition of sufficient capacity. For $S_1 \in \mathcal{P}_N$ as in (35) define $\mathcal{A}_1$ as

$$\mathcal{A}_1 = \{S \in \mathcal{P}_N \mid \sum_{i\in S} \underline{d}_i > \sum_{i\in S_1} \overline{d}_i\}$$

If $\mathcal{A}_1 = \phi$, let $S_2 \in \mathcal{P}_N$ be the set of subsystems which satisfies (33) for $D = \overline{D}$, then, since $S_2 \notin \mathcal{A}_1$,

$$\sum_{i\in S_2} \underline{d}_i \leq \sum_{i\in S_1} \overline{d}_i < \overline{D} \leq \sum_{i\in S_2} \overline{d}_i.$$

If $\mathcal{A}_1 \neq \phi$, since it is finite $\exists\ D_1 = \min\limits_{S\in\mathcal{A}_1}\{\sum_{i\in S} \underline{d}_i\}$ and $D_1 > \sum_{i\in S_1} \overline{d}_i$. Let $D_2 = \min\{D_1, \overline{D}\}$ and for $D_3 \in (\sum_{i\in S_1} \overline{d}_i, D_2)$ let $S_2 \in \mathcal{P}_N$ be the set for which (33) holds for $D = D_3$ and $S = S_2$. By lemma 3, $S_2 \notin \mathcal{A}_1$ giving

$$\sum_{i\in S_2} \underline{d}_i \leq \sum_{i\in S_1} \overline{d}_i < D_3 \leq \sum_{i\in S_2} \overline{d}_i.$$

Sufficiency. Assume (34) and (35) are true and let $D \in [\underline{D}, \overline{D}]$. If $D = \underline{D}$, (33) is satisfied for $S = Q$, as in (34). Suppose $D \in (\underline{D}, \overline{D}]$ and define the set $\mathcal{B}(D)$ as

$$\mathcal{B}(D) = \{S \in \mathcal{P}_N \mid \sum_{i\in S} \overline{d}_i < D\}.$$

If $Q \notin \mathcal{B}(D)$, (33) is satisfied for $S = Q$. If $Q \in \mathcal{B}(D)\ \exists\ S_1 \in \mathcal{B}(D) \ni$

$S_1 = \arg\{\max\limits_{S\in\mathcal{B}(D)} \sum_{i\in S} \overline{d}_i\}$ and

$$\underline{D} \leq \sum_{i\in Q} \overline{d}_i \leq \sum_{i\in S_1} \overline{d}_i < D \leq \overline{D}.$$

Therefore, by (35) $\exists\ S_2 \in \mathcal{P}_N \ni \sum_{i\in S_2} \underline{d}_i \leq \sum_{i\in S_1} \overline{d}_i < \sum_{i\in S_2} \overline{d}_i$. Hence, $S_2 \notin \mathcal{B}(D)$ and (33) is satisfied for $S = S_2$. Q.E.D.

For the subsequent work we make the following assumption concerning the cost functions $\varphi_i(d_i)$, i=1,...,N, defined by (6). For each $i\in\{1,...,N\}$ the function $\varphi_i(\cdot)$ is continuous and monotone increasing on the domain $[\underline{d}_i,\overline{d}_i]$. Certain properties of the optimal cost function, $C(D,\underline{v})$, are presented below as consequences of this assumption which will not be repeated in each lemma and theorem.

Lemma 5. Suppose $\underline{v}$ is feasible for D and $S=\mathscr{S}(\underline{v})$. For any $j\in S$, let $S_j = S-\{j\}$ and $\underline{v}_j = \mathscr{S}^{-1}[S_j]$. Then, if $C(\cdot,\underline{v}_j)$ is continuous on the domain for which $\underline{v}_j$ is feasible,

$$C(D,\underline{v}) = \min_{\underline{d}_j' \le x \le \overline{d}_j'} [C(D-x,\underline{v}_j)+\varphi_j(x)] \tag{36}$$

where

$$\underline{d}_j' = \max\{\underline{d}_j, D-\sum_{i\in S_j}\overline{d}_i\} \tag{37a}$$

and

$$\overline{d}_j' = \min\{\overline{d}_j, D-\sum_{i\in S_j}\underline{d}_i\}. \tag{37b}$$

Proof. From (37a,b) x is within the domain of $\varphi_j(\cdot)$. Furthermore, $\underline{v}_j$ is feasible $\forall$ D-x when $x\in[\underline{d}_j',\overline{d}_j']$ since

$$D-\sum_{i\in S_j}\overline{d}_i \le x \le D-\sum_{i\in S_j}\underline{d}_j <=> \sum_{i\in S_j}\underline{d}_j \le D-x \le \sum_{i\in S_j}\overline{d}_i.$$

Both $C(\cdot,\underline{v}_j)$ and $\varphi_j(\cdot)$ are continuous, hence, on the set $[\underline{d}_j',\overline{d}_j']$ the sum in (36) assumes its minimal value for some $x^*\in[\underline{d}_j',\overline{d}_j']$. From the definition of $C(D,\underline{v})$,

$$C(D,v) \le C(D-x^*,\underline{v}_j) + \varphi_j(x^*).$$

If $C(D,v)$ is strictly less than $C(D-x^*,\underline{v}_j)+\varphi_j(x^*)$, $\exists\{\hat{d}_i\}_{i\in S} \ni \underline{d}_i \le \hat{d}_i \le \overline{d}_i\ \forall i\in S$, $\sum_{i\in S}\hat{d}_i = D$, and

$$\sum_{i\in S}\varphi_i(\hat{d}_i) < C(D-x^*,\underline{v}_j)+\varphi_j(x^*).$$

Let $\hat{x} = \hat{d}_j$, then $\underline{d}_j \le \hat{x} \le \overline{d}_j$ and $D - \hat{x} = \sum_{i \in S_j} \hat{d}_i \rightarrow \sum_{i \in S_j} \underline{d}_i \le D - \hat{x} \le \sum_{i \in S_j} \overline{d}_i$ and hence $\hat{x} \in [\underline{d}_j', \overline{d}_j']$. Furthermore, $C(D-\hat{x}, \underline{v}_j) \le \sum_{i \in S_j} \hat{d}_i$, so we have

$$C(D-\hat{x}, \underline{v}_j) + \varphi_j(\hat{x}) < C(D-x^*, \underline{v}_j) + \varphi_j(x^*),$$

which contradicts the definition of x^*. Therefore,

$$C(D,v) = C(D-x^*, \underline{v}_j) + \varphi_j(x^*),$$

and (36) is proved.

Lemma 6. For $\underline{v} \in \{0,1\}^N$ and $S = \mathscr{S}(\underline{v})$, $C(\cdot, \underline{v})$ is monotone increasing and continuous on $[\sum_{i \in S} \underline{d}_i, \sum_{i \in S} \overline{d}_i]$.

Proof. Let $\|S\|$ denote the cardinality of the set S. Then, for $\|S\| = 1$ the lemma is true by the assumptions concerning the $\varphi_i(\cdot)$. Suppose the lemma is true for all $C(\cdot, \underline{v})$ when $\|\mathscr{S}(\underline{v})\| \le k$ for some $k \in \{1, \ldots, N-1\}$, but that for $S_{k+1} \in \mathscr{P}_N$ where $\|S_{k+1}\| = k+1$, $C(\cdot, \underline{v}_{k+1})$ is not monotone increasing on $[\sum_{i \in S_{k+1}} \underline{d}_i, \sum_{i \in S_{k+1}} \overline{d}_i]$ where $\underline{v}_{k+1} = \mathscr{S}^{-1}(S_{k+1})$. Then $\exists\, D_1, D_2 \in [\sum_{i \in S_{k+1}} \underline{d}_i, \sum_{i \in S_{k+1}} \overline{d}_i] \ni D_1 < D_2$ and $C(D_1, \underline{v}_{k+1}) > C(D_2, \underline{v}_{k+1})$.

For any $j \in S_{k+1}$, define $S_k = S_{k+1} - \{j\}$ and $\underline{v}_k = \mathscr{S}^{-1}[S_k]$. Then $\|S_k\| = k \ge 1$ and by the induction hypothesis, $C(\cdot, \underline{v}_k)$ is monotone increasing and continuous on $[\sum_{i \in S_k} \underline{d}_i, \sum_{i \in S_k} \overline{d}_i]$. Define, for $\ell = 1,2$

$$\underline{d}_\ell = \max\{\underline{d}_j, D_\ell - \sum_{i \in S_k} \overline{d}_i\} \text{ and } \overline{d}_\ell' = \min\{\overline{d}_i, D_\ell - \sum_{i \in S_k} \underline{d}_i\},$$

Lemma 5 implies $\exists$ for $\ell = 1,2$, $x_\ell \in [\underline{d}_\ell', \overline{d}_\ell'] \ni C(D_\ell, \underline{v}_{k+1}) = C(D_\ell - x_\ell, \underline{v}_k) + \varphi_j(x_\ell)$

$= \min_{\underline{d}_\ell' \le x \le \overline{d}_\ell'} [C(D_\ell - x, \underline{v}_k) + \varphi_j(x)]$. Hence, $C(D_1 - x_1, \underline{v}_k) + \varphi_j(x_1) > C(D_2 - x_2, \underline{v}_k) + \varphi_j(x_2)$. Consider the following two cases:

Case 1: $D_1 \geq x_2 + \sum_{i\in S_k} \underline{d}_i$. Then $D_1 < D_2$ implies $\sum_{i\in S_k} \underline{d}_i \leq D_1 - x_2 < D_2 - x_2$ and hence $\underline{v}_k$ is feasible for $D_1 - x_2$. By the monotonicity of $C(\cdot, \underline{v}_k)$, $C(D_1-x_2,\underline{v}_k) + \varphi_j(x_2) \leq C(D_2-x_2,\underline{v}_k) + \varphi_j(x_2) < C(D_1-x_1,\underline{v}_k) + \varphi_j(x_1)$, which contradicts the definition of x_1.

Case 2: $D_1 < x_2 + \sum_{i\in S_k} \underline{d}_i$. Let $\tilde{D} = \sum_{i\in S_k} \underline{d}_i$ and $\tilde{x} = D_1 - \sum_{i\in S_k} \underline{d}_i$. Then $\underline{v}_k$ is feasible for $\tilde{D}$ and

$$C(\tilde{D},\underline{v}_k) \leq C(D_2-x_2,\underline{v}_k).$$

Furthermore,

$$\underline{d}_j \leq \underline{d}'_1 \leq \overline{d}'_1 \leq D_1 - \sum_{i\in S_k} \underline{d}_i = \tilde{x} < x_2 \leq \overline{d}_j,$$

hence, $\tilde{x}$ is in the domain of $\varphi_j(\cdot)$ which implies $\varphi_j(\tilde{x}) \leq \varphi_j(x_2)$. Since $\tilde{D} = D_1 - \tilde{x}$ we have

$$C(D_1-\tilde{x},\underline{v}_k) + \varphi_j(\tilde{x}) \leq C(D_2-x_2,\underline{v}_k) + \varphi_j(x_2) < C(D_1-x_1,\underline{v}_k) + \varphi_j(x_1),$$

which also contradicts the definition of x_1.

Therefore, $C(\cdot,\underline{v}_{k+1})$ is monotone increasing. Right continuity of $C(\cdot,\underline{v}_{k+1})$ is proved as follows. Let $\hat{D} \in [\sum_{i\in S_{k+1}} \underline{d}_i, \sum_{i\in S_{k+1}} \overline{d}_i)$ and define $\underline{d}' = \max\{\underline{d}_j, \hat{D} - \sum_{i\in S_k} \overline{d}_i\}$ and $\overline{d}' = \min\{\overline{d}_j, \hat{D} - \sum_{i\in S_k} \underline{d}_i\}$. By lemma 5, $\exists\, \hat{x} \in [\underline{d}', \overline{d}'] \ni$

$$C(\hat{D},\underline{v}_{k+1}) = C(\hat{D} - \hat{x},\underline{v}_k) + \varphi_j(\hat{x}).$$

If $\hat{x} < \overline{d}'$, let $\delta_1 = \min\{\overline{d}' - \hat{x}, \sum_{i\in S_{k+1}} \overline{d}_i - \hat{D}\}$. Then for any $\varepsilon \ni 0 < \varepsilon < \delta_1$, $\hat{x} + \varepsilon$ is in the domain of $\varphi_j(\cdot)$ and $\hat{D} + \varepsilon$ is in the domain of $C(\cdot,\underline{v}_{k+1})$. Furthermore, $\varphi_j(\hat{x}+\varepsilon) \geq \varphi_j(\hat{x})$ and by definition,

$$C(\hat{D}+\varepsilon,\underline{v}_{k+1}) \leq C[(\hat{D}+\varepsilon)-(\hat{x}+\varepsilon),\underline{v}_k] + \varphi_j(\hat{x}+\varepsilon).$$

Therefore,

$$0 \leq C(\hat{D}+\varepsilon,\underline{v}_{k+1}) - C(\hat{D},\underline{v}_{k+1}) \leq C(\hat{D}-\hat{x},\underline{v}_k) + \varphi_j(\hat{x}+\varepsilon) - C(\hat{D}-\hat{x},v_k) + \varphi_j(\hat{x})$$

$$= \varphi_j(\hat{x}+\varepsilon) - \varphi_j(\hat{x})$$

and by continuity of $\varphi_j(\cdot)$, the last term goes to zero with ε. If $\hat{x}=\overline{d}'$, let $\delta_2 = \min\{\sum_{i\in S_{k+1}} \overline{d}_i - \hat{D}, \sum_{i\in S_k} \overline{d}_i + \overline{d}' - \hat{D}\}$. Note that if $\overline{d}' = d_j$, $\sum_{i\in S_k} \overline{d}_i + \overline{d}' - \hat{D} = \sum_{i\in S_{k+1}} \overline{d}_i - \hat{D} > 0$, and if $\overline{d}' = \hat{D} - \sum_{i\in S_k} \underline{d}_i$, $\sum_{i\in S_k} \overline{d}_i + \hat{D} - \sum_{i\in S_k} \underline{d}_i - \hat{D} = \sum_{i\in S_k} \overline{d}_i - \sum_{i\in S_k} \underline{d}_i > 0$, therefore, $\delta_2 > 0$ and for any $\varepsilon \ni 0 < \varepsilon < \delta_2$, $\hat{D}+\varepsilon$ is in the domain of $C(\cdot,\underline{v}_{k+1})$. Furthermore, $\varepsilon \leq \sum_{i\in S_k} \overline{d}_i + \overline{d}' - \hat{D} \rightarrow \sum_{i\in S_k} \overline{d}_i \geq \hat{D} + \varepsilon - \overline{d}'$, hence $\underline{v}_k$ is feasible for $\hat{D}+\varepsilon-\hat{x}$ and by definition of $C(\cdot,\underline{v}_{k+1})$, $C(\hat{D}+\varepsilon,\underline{v}_{k+1}) \leq C(\hat{D}+\varepsilon-\hat{x},\underline{v}_k) + \varphi_j(\hat{x})$. Therefore,

$$0 \leq C(\hat{D}+\varepsilon,\underline{v}_{k+1}) - C(\hat{D},\underline{v}_{k+1}) \leq C(\hat{D}+\varepsilon-\hat{x},\underline{v}_k) + \varphi_j(\hat{x}) - C(\hat{D}-\hat{x},\underline{v}_k) - \varphi_j(\hat{x})$$

$$= C(\hat{D}+\varepsilon-\hat{x},\underline{v}_k) - C(\hat{D}-\hat{x},\underline{v}_k)$$

and the last term goes to zero with ε by the induction hypothesis. Therefore, $C(\cdot,\underline{v}_{k+1})$ is right continuous.

To prove left continuity, suppose $\tilde{D} \in (\sum_{i\in S_{k+1}} \underline{d}_i, \sum_{i\in S_{k+1}} \overline{d}^i]$ and let $\{D_n\}_{i=1}^{\infty} \subset [\sum_{i\in S_{k+1}} \underline{d}_i, \sum_{i\in S_{k+1}} \overline{d}^i]$ be any sequence such that $D_n \uparrow \tilde{D}$. For each n define $\underline{d}'_n$, and $\overline{d}'_n$ as $\underline{d}'_n = \max\{\underline{d}_j, D_n - \sum_{i\in S_k} \overline{d}_i\}$ and $\overline{d}'_n = \min\{\underline{d}_j, D_n - \sum_{i\in S_k} \underline{d}_i\}$. By lemma 5 $\exists$ $x_n \in [\underline{d}'_n, \overline{d}'_n]$ for each n such that

$$C(D_n,\underline{v}_{k+1}) = C(D_n - x_n,\underline{v}_k) + \varphi_j(x_n).$$

Define $\underline{d}' = \max\{\underline{d}_j, \tilde{D} - \sum_{i\in S_k} \overline{d}_i\}$ and $\overline{d}' = \min\{\underline{d}_j, \tilde{D} - \sum_{i\in S_k} \underline{d}_i\}$ and for each n define $\hat{x}_n$ as follows.

$$\hat{x}_n = \begin{cases} \tilde{D} - D_n + x_n & \text{if } \underline{d}' < \tilde{D} - D_n + x_n < \overline{d}' \\ \underline{d}' & \text{if } \tilde{D} - D_n + x_n \leq \underline{d}' \\ \overline{d}' & \text{if } \tilde{D} - D_n + x_n \geq \overline{d}' \end{cases}$$

Then for each n, $0 \le \hat{x}_n - x_n \le \tilde{D} - D_n$. This is shown by considering each of the three cases defining $\hat{x}_n$.

Case 1. $\hat{x}_n = \tilde{D} - D_n + \hat{x}_n = \tilde{D} - D_n \ge 0$

Case 2. $\hat{x}_n = \underline{d}'$ and $\tilde{D} - D_n + x_n \le \underline{d}'$. From the definitions of $\underline{d}'_n$, $\underline{d}'$ and the fact $\tilde{D} - D_n \ge 0$ we have

$$\underline{d}'_n \le \tilde{D} - D_n + x_n \le \underline{d}'$$

$$\rightarrow \underline{d}' = \tilde{D} - \sum_{i \in S_k} \overline{d}_i > \max\{\underline{d}_j, D_n - \sum_{i \in S_k} \overline{d}_i\}$$

$$\therefore \tilde{D} - D_n \ge \underline{d}' - \underline{d}'_n \ge \underline{d}' - x_n \ge 0, \text{ since } x_n \ge \underline{d}'_n.$$

Case 3. $\hat{x}_n = \overline{d}'$ and $\tilde{D} - D_n + x_n \ge \overline{d}'$. This implies directly $\tilde{D} - D_n \ge \overline{d}' - x_n = \hat{x}_n$, and since by definition, $\overline{d}' \ge \overline{d}_n$, then $\overline{d}' - x_n \ge \overline{d}_n - x_n \ge 0$.

Therefore, as shown in lemma 5, $\hat{x}_n$ is in the domain of $\varphi_j(\cdot)$ and $\underline{v}_k$ is feasible for $\tilde{D} - \hat{x}_n$ for all n since $\hat{x}_n \in [\underline{d}', \overline{d}']$. Hence, for every n,

$$0 \le C(\tilde{D}, \underline{v}_{k+1}) - C(D_n, \underline{v}_{k+1}) \pm [C(\tilde{D} - \hat{x}_n, \underline{v}_k) + \varphi_j(\hat{x}_n)] = C(\tilde{D}, \underline{v}_{k+1}) - C(\tilde{D} - \hat{x}_n, \underline{v}_k) - \varphi_j(\hat{x}_n)$$
$$+ C(\tilde{D} - \hat{x}_n, \underline{v}_k) - C(D_n - x_n, \underline{v}_k) + \varphi_j(\hat{x}_n) - \varphi_j(x_n)$$
$$\le C(\tilde{D} - \hat{x}_n, \underline{v}_k) - C(D_n - x_n, \underline{v}_k) + \varphi_j(\hat{x}_n) - \varphi_j(x_n),$$

where the last inequality results from lemma 5.

Since both $C(\cdot, \underline{v}_k)$ and $\varphi_j(\cdot)$ are continuous on closed intervals, they are uniformly continuous. Hence, for any $\varepsilon > 0$ $\exists N \ni \forall n > N$ the last term is smaller than ε. Therefore, $C(\tilde{D}, \underline{v}_{k+1}) - C(D_n, \underline{v}_{k+1}) \rightarrow 0$ and the lemma is proved.

The following result is based upon the assumption of convexity for the $\varphi_i(\cdot)$, i=1,...,N, rather than continuity and monotonicity. Convexity, of course, implies continuity. Since convexity is too strong of an assumption for the applications to be considered, this result will not be used in the sequel and is included merely to indicate an alternative development.

Lemma 7. If $\varphi_i(\cdot)$ is convex on $[\underline{d}_i,\overline{d}_i]$ for $i=1,\ldots,N$, then for any $\underline{v}\in\{0,1\}^N$ and $S=\mathscr{S}(\underline{v})$, $C(\cdot,\underline{v})$ is convex on $[\sum_{i\in S}\underline{d}_i,\sum_{i\in S}\overline{d}_i]$.

Proof. If $|S|=1$ the lemma is trivially true. Suppose for some $k\in\{1,\ldots,N-1\}$ the lemma holds if $|S|\leq k$ but $\exists\ S_{k+1}\in\mathscr{P}_N \ni |S_{k+1}|=k+1$ and for $\underline{v}_{k+1}=\mathscr{S}^{-1}(S_{k+1})$ $C(\cdot,\underline{v}_{k+1})$ is not convex. Then, $\exists\ D_1, D_2, D_3$, all in $[\sum_{i\in S_{k+1}}\underline{d}_i,\sum_{i\in S_{k+1}}\overline{d}_i]$ with $D_1<D_3<D_2 \ni$

$$C(D_3,\underline{v}_{k+1}) > \alpha\ C(D_1,\underline{v}_{k+1}) + (1-\alpha)C(D_2,\underline{v}_{k+1})$$

where

$$\alpha = \frac{D_3 - D_1}{D_2 - D_1}.$$

For $j\in S_{k+1}$ define $S_k=S_{k+1}-\{j\}$ and let $\underline{v}_k=\mathscr{S}^{-1}(S_k)$. Since, by the induction hypothesis, $C(\cdot,\underline{v}_k)$ is convex, it is also continuous as is $\varphi_j(\cdot)$, therefore, by lemma 5, $\exists$ for $\ell=1,2$, $x_\ell\in[\underline{d}_j,\overline{d}_j]\ni$

$$C(D_\ell,\underline{v}_{k+1}) = C(D_\ell - x_\ell,\underline{v}_k) + \varphi_j(x_\ell).$$

Then

$$C(D_3,\underline{v}_{k+1}) > \alpha[C(D_1-x_1,\underline{v}_k)+\varphi_j(x_1)] + (1-\alpha)[C(D_2-x_2,\underline{v}_k)+\varphi_j(x_2)]$$

and by convexity of $C(\cdot,\underline{v}_k)$ and $\varphi_j(\cdot)$, we have

$$C(D_3,\underline{v}_{k+1}) > C[D_3 - (\alpha x_1 + (1-\alpha)x_2),\underline{v}_k] + \varphi_j[\alpha x_1 + (1-\alpha)x_2],$$

which contradicts the definition of $C(\cdot,\underline{v}_{k+1})$ as the minimal cost in view of lemma 5. $\therefore C(\cdot,\underline{v}_{k+1})$ is convex.

We resume now with the assumption that the $\varphi_i(\cdot)$ are monotone increasing and continuous. Now, assuming a set of N subsystems has sufficient capacity for a given range of demand, $[\underline{D},\overline{D}]$, we wish to consider the minimal cost possible using any feasible $\underline{v} \in \{0,1\}^N$ for a given demand $D \in [\underline{D},\overline{D}]$. If we define the set of feasible $\underline{v}$ for a given demand D, $\mathscr{V}(D)$, as

$$\mathscr{V}(D) = \{\underline{v} \in \{0,1\}^N \mid \underline{v} \text{ if feasible for } D \in [\underline{D},\overline{D}]\} \tag{38}$$

then the minimal cost for meeting D with any collection of subsystems, $C^*(D)$, is defined by

$$C^*(D) = \min_{\underline{v}\in\mathscr{V}(D)} C(D,\underline{v}), \quad D\in[\underline{D},\overline{D}]. \tag{39}$$

The following results characterize the function $C^*(\cdot)$ which is not necessarily monotone increasing or continuous on all of $[\underline{D},\overline{D}]$.

<u>Lemma 8</u>. If for the interval $(D_1,D_2) \subset [\underline{D},\overline{D}]$ $\forall D\in(D_1,D_2)$ $\exists \underline{v}_D \in \mathscr{V}(D) \ni$

$$C^*(D) = C(D,\underline{v}_D) \tag{40}$$

and for $S_D = \mathscr{I}(v_D)$,

$$(D_1,D_2) \subset [\sum_{i\in S_D} \underline{d}_i, \sum_{i\in S_D} \overline{d}_i], \tag{41}$$

then $C^*(\cdot)$ is monotone increasing and continuous on (D_1,D_2).

<u>Proof</u>. For any D', $D''\in(D_1,D_2) \ni D' < D''$, let $\underline{v}_{D'}$, $\underline{v}_{D''}$ be the vectors for which (40) holds for D' and D'' respectively, and let $S_{D'} = \mathscr{I}(\underline{v}_{D'})$ and $S_{D''} = \mathscr{I}(\underline{v}_{D''})$ be the corresponding sets for which (41) is satisfied. Then $\underline{v}_{D'}$ and $\underline{v}_{D''}$ are both feasible for all $D \in (D_1,D_2)$ and $C(\cdot,\underline{v}_{D'})$, $C(\cdot,\underline{v}_{D''})$ are monotone increasing on this interval. Therefore,

$$C^*(D') = C(D',\underline{v}_{D'}) \leq C(D',\underline{v}_{D''}) \leq C(D'',\underline{v}_{D''}) = C^*(D'')$$

implying the monotonicity of $C^*(\cdot)$ on (D_1,D_2).

Now, for $\tilde{D}\in(D_1,D_2)$ let $\underline{v}_{\tilde{D}}$ and $S_{\tilde{D}} = \mathscr{I}(\underline{v}_{\tilde{D}})$ be the corresponding vector and set for which (40) and (41) hold, respectively. Then, from the definition of $C^*(\cdot)$,

$$0 \le C^*(D) - C^*(\tilde{D}) \le C(D,\underline{v}_{\tilde{D}}) - C(\tilde{D},\underline{v}_{\tilde{D}}) \ \forall D \in (\tilde{D},D_2)$$

and the last term goes to zero as $D \downarrow \tilde{D}$ by the continuity of $C(\cdot,\underline{v}_{\tilde{D}})$. Hence, $C^*(\cdot)$ is right continuous on (D_1,D_2).

For every $D \in (D_1,\tilde{D})$ let $\underline{v}_D$ and $S_D = \mathscr{S}(\underline{v}_D)$ be the corresponding vector and set for which (40) and (41) hold, respectively. From the definition of $C^*(\cdot)$ and its monotonicity on (D_1,D_2),

$$0 \le C^*(\tilde{D}) - C^*(D) \le C(\tilde{D},\underline{v}_D) - C(D,\underline{v}_D) \ \forall D \in (D_1,\tilde{D}).$$

Now, since $\{0,1\}^N$ is finite and $C(\cdot,\underline{v}_D)$ is continuous $\forall D \in (D_1,\tilde{D})$, for any $\varepsilon > 0 \ \exists\ \delta_\varepsilon > 0 \ \exists\ \delta_\varepsilon > 0 \ni \forall\ 0 < \delta < \delta_\varepsilon$, and $\forall\ D \in (D_1,\tilde{D})$, $C(\tilde{D},\underline{v}_D) - C(\tilde{D}-\delta,\underline{v}_D) < \varepsilon$. Therefore, the last term above goes to zero as $D \uparrow \tilde{D}$, and hence, $C^*(\cdot)$ is continuous on (D_1,D_2).

<u>Lemma 9</u>. For $C^*(\cdot)$ defined in (39),

$$\lim_{D \to \tilde{D}^-} C^*(D) = C^*(\tilde{D}^-) \text{ exists } \forall\ \tilde{D} \in (\underline{D},\overline{D}] \tag{42}$$

and

$$\lim_{D \to \tilde{D}^+} \overset{*}{C}(D) = C^*(\tilde{D}^+) \text{ exists } \forall\ \tilde{D} \in [\underline{D},\overline{D}). \tag{43}$$

<u>Proof</u>. For any $\tilde{D} \in (\underline{D},\overline{D}]\ \exists\ D' \in (D_1,\tilde{D}) \ni$ the interval $(D',\tilde{D})$ satisfies the hypothesis of lemma 8 since $\{0,1\}^N$ is finite. Since $C^*(\cdot)\uparrow$ and is continuous on $(D',\tilde{D})$, the limit $C^*(\tilde{D}^-)$ exists in (42). The result (43) is proved by a similar arguement.

<u>Lemma 10</u>. $\exists\ D_1,\ D_2 \in [\underline{D},\overline{D}] \ni$

$$D_1 < D_2 \quad \text{and} \tag{44}$$

$$C^*(D_1) > C^*(D_2), \tag{45}$$

if $\exists\ \tilde{D} \in [\underline{D}, \bar{D}] \ni$

$$D_1 < \tilde{D} \leq D_2 \tag{46}$$

and $\exists\ \tilde{S} \in \mathscr{P}_N \ni$

$$\sum_{i \in \tilde{S}} \underline{d}_i = \tilde{D} \tag{47}$$

and for $\underline{\tilde{v}} \in \mathscr{S}^{-1}[\tilde{S}]$,

$$C^*(\tilde{D}) = C(\tilde{D}, \underline{\tilde{v}}) \tag{48}$$

and

$$C^*(\tilde{D}^-) > C^*(\tilde{D}). \tag{49}$$

Proof. Sufficiency. Let $\tilde{D}$, $\tilde{S}$ and $\underline{\tilde{v}}$ satisfy (46)-(49) and let $D_2 = \tilde{D}$. Then by (49) $\exists\ D_1$ such that (44) and (45) are satisfied.

Necessity. Let D_1, D_2 satisfy (44) and (45) and define $\underline{v}_o^o \in \{0,1\}^N$ as the vector for which $C^*(D_2) = C(D_2, \underline{v}_o^o)$ and let $S_o^o = \mathscr{S}(\underline{v}_o^o)$. Define $D_o^o = \sum_{i \in S_o^o} \underline{d}_i$. If $D_o^o \leq D_1$ we have,

$$C^*(D_1) \leq C(D_1, \underline{v}_o^o) \leq C(D_2, \underline{v}_o^o) = C^*(D_2)$$

which contradicts (45). Hence $D_1 < D_o^o \leq D_2$. If (48) and (49) are satisfied for $\tilde{D} = D_o^o$, $\underline{\tilde{v}} = \underline{v}_o^o$, the lemma is proved. If (48) is not satisfied, proceed with step A below, if (48) is satisfied, but (49) is not satisfied, proceed with step B.

Step A. (48) does not hold for $\underline{\tilde{v}} = \underline{v}_j^k$, $\tilde{D} = D_j^k$. Hence $\exists\ \underline{v}_j^{k+1} \in \{0,1\}^N \ni$

$$C^*(D_j^k) = C(D_j^k, \underline{v}_j^{k+1}) < C(D_j^k, \underline{v}_j^k). \tag{50}$$

Let $S_j^{k+1} = \mathscr{S}(\underline{v}_j^{k+1})$ and define $D_j^{k+1} = \sum_{i \in S_j^{k+1}} \underline{d}_i$.

If (48) and (49) hold for $\underline{\tilde{v}} = \underline{\tilde{v}}_j^{k+1}$ and $\tilde{D} = D_j^{k+1}$, then, as is shown below, the lemma is proved. Otherwise, if (48) does not hold, repeat step A. If (48) holds but (49) does not hold, define $K_j = k+1$ and proceed with step B.

Step B. (49) does not hold for $\tilde{D} = D_j^{k_j}$ but (48) holds for $\tilde{\underline{v}} = \underline{v}_j^{k+1}$. It is shown below that $D_1 < D_j^{kj}$, hence $\exists\ \hat{D}_{j+1} \in (D_1, D_j^{kj}) \ni$

$$C^*(\hat{\hat{D}}_{j+1}) \leq C^*(D_j^{k_j}). \tag{51}$$

This follows from lemma 8. Let $\underline{v}_{j+1}^o \in \{0,1\}^N$ be the vector $\ni C^*(\hat{D}_{j+1})$ $= C(\hat{D}_{j+1}, \underline{v}_{j+1}^o)$ and define $S_{j+1}^o = \mathscr{S}(\underline{v}_{j+1}^o)$, and $D_{j+1}^o = \sum_{i \in S_{j+1}^o} \underline{d}_i$.

It is shown below that if (48) and (49) hold for $\tilde{D} = D_{j+1}^o$ and $\tilde{\underline{v}} = \underline{v}_{j+1}^o$, the lemma is proved. Otherwise, proceed with Step A if (48) does not hold, and with Step B if (48) holds but (49) does not.

We first show that (46) is satisfied by $\tilde{D} = D_j^k$ for D_j^k defined above. Clearly, $D_j^k \leq D_2$. It was shown above that $D_1 < D_o^o$ and $C(D_o^o) \leq C^*(D_2)$. Assume that for D_j^k, $D_1 < D_j^k$ and $C(D_j^k) \leq C^*(D_2)$.

If (48) does not hold, it is clear from Step A that $D_j^{k+1} \leq D_j^k$ which implies $C(D_j^{k+1}, \underline{v}_j^{k+1}) \leq C(D_j^k, \underline{v}_j^{k+1})$. Then, from (50), $C(D_j^{k+1}, \underline{v}_j^{k+1}) \leq C^*(D_2)$. Now, if $D_j^{k+1} \leq D_1$, then $\underline{v}_j^{k+1}$ is feasible for D_1 and we have

$$C^*(D_1) \leq C(D_1, \underline{v}_j^{k+1}) \leq C(D_j^k, \underline{v}_j^{k+1}) \leq C(D_2)$$

which contradicts (45), therefore $D_1 < D_j^{k+1}$. Suppose $D_{j+1}^o \leq D_1$, then since $D_{j+1} > D_1$, by definition, $\underline{v}_{j+1}^o$ is feasible for D_1 and we have

$$C^*(D_1) \leq C(D_1, \underline{v}_{j+1}^o) \leq C(\hat{D}_{j+1}, \underline{v}_{j+1}^o).$$

But from (51) and the hypothesis $C^*(D_j^k) \leq C^*(D_2)$, we have $C^*(D_1) \leq C^*(\hat{D}_{j+1})$ $\leq C^*(D_j^{k_j}) \leq C^*(D_2)$, which contradicts (45). Therefore, D_j^k defined by Step A and B all satisfy (46) and by definition of S_j^k, (47) is also satisfied for $\tilde{S} = S_j^k$. Finally, it is easily seen that at each step, S_j^{k+1} or S_{j+1}^o can not equal some previously defined S_m^n. Therefore, since $\mathscr{P}_N$ is finite, the algorithm defined above must terminate for some D_k^j and $\underline{v}_j^k$ which satisfy (48) and (49), thus proving the lemma.

Lemma 11. If $C^*(\cdot)$ is continuous on an interval, $[D_1,D_2] \subset [\underline{D},\overline{D}]$, it is monotone increasing on that interval.

Proof. This is a direct implication of the previous lemma since if $\exists$ D', D" $\in [D_1,D_2] \ni D' < D''$ and $C^*(D') > C^*(D'')$, there is a discontinuity between D' and D" of the type (49).

Lemma 12. If for $\tilde{D} \in (\underline{D},\overline{D}]$ $\exists \underline{\tilde{v}} \in \{0,1\}^N$ and $\tilde{S} = \mathscr{S}(\underline{\tilde{v}}) \ni$

$$C^*(\tilde{D}) = C(\tilde{D},\underline{\tilde{v}}) \tag{52}$$

and

$$\sum_{i \in \tilde{S}} \underline{d}_i < \tilde{D} \leq \sum_{i \in \tilde{S}} \overline{d}_i \tag{53}$$

then

$$C^*(\tilde{D}) = C^*(\tilde{D}^-). \tag{54}$$

Proof. Suppose for $\tilde{D} \in (\underline{D},\overline{D}]$, $\underline{\tilde{v}}$ and $\tilde{S} = \mathscr{S}(\underline{\tilde{v}})$, (52) and (53) are satisfied. Then for $D \in [\sum_{i \in \tilde{S}} \underline{d}_i, \tilde{D})$,

$$C^*(D) \leq C(D,\underline{\tilde{v}}) \leq C(\tilde{D},\underline{\tilde{v}}) = C^*(\tilde{D}).$$

$\therefore C^*(\tilde{D}^-) \leq C^*(\tilde{D})$. Suppose $C^*(\tilde{D}) - C^*(\tilde{D}^-) = \alpha > 0$. Clearly $\exists$ $D' \in [\underline{D},\tilde{D}) \ni$ for the interval $(D',\tilde{D})$ the hypothesis of lemma 8 are satisfied and hence $C^*(\cdot)$ is monotone increasing and continuous on $(D',\tilde{D})$. Then,

$$C^*(\tilde{D}) - C^*(D) \geq \alpha \quad \forall D \in (D',\tilde{D}). \tag{55}$$

Let $\underline{v}_D$ be the vector for which $C^*(D) = C(D,\underline{v}_D)$ for $D \in (D',\tilde{D})$. By the choice of D' we can select v_D so that it is feasible on the entire interval $[D',\tilde{D}]$. By the continuity of $C(\cdot,\underline{v})$ for each $\underline{v} \in \{0,1\}^N$, there exists $\delta_\alpha > 0$ such that

$$0 \le C(\tilde{D},\underline{v}_D) - C(\tilde{D}-\delta,\underline{v}_D) \le \alpha/2 \text{ when } 0<\delta\le\delta_\alpha.$$

But, $C^*(\tilde{D}) \le C(\tilde{D},\underline{v}_D)\ \forall\ D\in[D',\tilde{D}]$, so for $D=\tilde{D}-\delta$ where $0<\delta\le\min\{\delta_\alpha,\tilde{D}-D'\}$ we have

$$0\le C^*(\tilde{D}) - C^*(D) \le C(\tilde{D},\underline{v}_D) - C(D,\underline{v}_D) \le \alpha/2$$

which contradicts (55). Therefore, $\alpha=0$ and (54) is proved.

Lemma 13. If for $\tilde{D}\in[\underline{D},\overline{D})\ \exists\ \underline{\tilde{v}}\in\{0,1\}^N$ and $\tilde{S}=\mathscr{S}(\underline{\tilde{v}})\ni$

$$C^*(\tilde{D}) = C(\tilde{D},\underline{\tilde{v}}) \tag{56}$$

and

$$\sum_{i\in\tilde{S}} \underline{d}_i \le \tilde{D} < \sum_{i\in\tilde{S}} \overline{d}_i \tag{57}$$

then

$$C^*(\tilde{D}) = C^*(\tilde{D}^+) \tag{58}$$

Proof. For $\tilde{D}$, $\tilde{S}$, $\underline{\tilde{v}}$ defined above assume (56) and (57) hold. $\exists\ D'\in(\tilde{D},\overline{D}]$ such that the hypothesis of lemma 8 are satisfied for the interval $(\tilde{D},D')$. Hence, $C^*(\cdot)$ is monotone increasing and continuous on $(\tilde{D},D')$. Furthermore, for each $D\in(\tilde{D},D')\ \exists\ \underline{v}_D\in\{0,1\}^N\ni C^*(D)=C(D,\underline{v}_D)$ and $\underline{v}_D$ is feasible on the entire interval $[\tilde{D},D']$. Therefore,

$$C^*(\tilde{D}) \le C(\tilde{D},\underline{v}_D) \le C(D,\underline{v}_D) = C^*(D)\quad \forall D\in(\tilde{D},D')$$

which implies $C^*(\tilde{D})\le C^*(\tilde{D}^+)$.

Now, $\forall D\in[\tilde{D},\sum_{i\in\tilde{S}}\overline{d}_i]$, $C^*(D)\le C(D,\underline{\tilde{v}})$ and since $C(D,\underline{\tilde{v}})\downarrow C^*(\tilde{D})$ as $D\downarrow\tilde{D}$ we have $C^*(\tilde{D}^+)\le C^*(\tilde{D})$. Therefore $C^*(\tilde{D})=C^*(\tilde{D}^+)$, proving (58).

The following theorem incorporates the above results to characterize $C^*(\cdot)$ on the entire range $[\underline{D},\overline{D}]$.

Theorem 3. If $C^*(\cdot)$ is discontinuous at $\tilde{D} \in [\underline{D},\overline{D}]$ then the discontinuity is one of the following two types.

Type 1. $\exists\ S_1 \in \mathscr{P}_N$ with $\underline{v}_1 = \mathscr{S}^{-1}(S_1) \ni$

$$\tilde{D} = \sum_{i\in S_1} \underline{d}_i, \tag{59}$$

$$C^*(\tilde{D}) = C(\tilde{D},\underline{v}_1), \tag{60}$$

and

$$C^*(\tilde{D}^-) > C^*(\tilde{D}). \tag{61}$$

Type 2. $\exists\ S_2 \in \mathscr{P}_N$ with $\underline{v}_2 = \mathscr{S}^{-1}(S_2) \ni$

$$\tilde{D} = \sum_{i\in S_2} \overline{d}_i, \tag{62}$$

$$C^*(\tilde{D}) = C(\tilde{D},\underline{v}_2) \tag{63}$$

and

$$C^*(\tilde{D}^+) > C^*(\tilde{D}). \tag{64}$$

Furthermore, $C^*(\cdot)$ is monotone increasing on any interval in $[\underline{D},\overline{D}]$ which contains no points of discontinuity of $C^*(\cdot)$.

Proof. The final statement is merely lemma 11. Suppose $\tilde{D} \in (\underline{D},\overline{D})$ is a point of discontinuity but neither (59) and (60) or (62) and (63) hold at $\tilde{D}$. Then clearly $\tilde{D}$ is contained in an interval which satisfies the hypothesis of lemma 8, which implies $C(\cdot)$ is continuous at $\tilde{D}$. Therefore, if $\tilde{D} \in (\underline{D},\overline{D})$ it satisfies either (59) and (60) or (62) and (63). Choose $\varepsilon > 0$ such that $\tilde{D}$ is the only points of discontinuity on $(\tilde{D} - \varepsilon, \tilde{D} + \varepsilon)$. This is possible since $\exists\ \varepsilon_1, \varepsilon_2 > 0 \ni$ the intervals $(\tilde{D} - \varepsilon_1, \tilde{D})$ and $(\tilde{D}, \tilde{D} + \varepsilon_2)$ satisfy the hypothesis of lemma 8, therefore implying $C^*(\cdot)$ is monotone increasing and continuous on these intervals. If

$C^*(\tilde{D}^-) > C^*(\tilde{D}^+)$, lemma 10 implies (59) and (60) are satisfied at $\tilde{D}$ and by lemma 13, $C^*(\tilde{D}^+) = C^*(\tilde{D})$, so (61) is also satisfied.

If $C^*(\tilde{D}^-) < C^*(\tilde{D}^+)$, the discontinuity cannot satisfy (59)-(61) simultaneously since lemma 13 implies that if $\tilde{D}$ satisfies (59) and (60), $C^*(D) = C^*(D^+)$, so (61) cannot be satisfied. Suppose (62) and (63) are satisfied. Then lemma 12 implies $C^*(\tilde{D}^-) = C^*(D)$, therefore the discontinuity is of type 2.

Finally, if $\tilde{D} = \underline{D}$, then only $C^*(\tilde{D}^+)$ is defined. Suppose $C^*(\tilde{D}^+) < C^*(\underline{D})$, then choose $\varepsilon \ni C^*(\cdot)$ is continuous on $(\underline{D}, \underline{D} + \varepsilon)$ and $\exists D' \in (\underline{D}, \underline{D} + \varepsilon) \ni C^*(\underline{D}) > C^*(D')$. Then, by lemma 10 $\exists D'' \in (\underline{D}, D')$ which is a point of discontinuity for $C^*(\cdot)$, but this contradicts the definition of ε since $D'' < \underline{D} + \varepsilon$. Therefore, $C^*(\tilde{D}^+) > C^*(\underline{D})$ and hence the discontinuity must be of Type 2 since if $\tilde{D} < \sum_{i \in S} \bar{d}_i \ \forall S \ni$ for $\underline{v} = \mathscr{S}^{-1}(S)$, $C^*(\tilde{D}) = C(\tilde{D}, \underline{v})$, $C^*(\tilde{D}^+) = C^*(\tilde{D})$ by lemma 13. The fact that if $\tilde{D} = \bar{D}$ then the discontinuity is of type 1 is shown by assuming $C^*(\tilde{D}^{\theta}) < C^*(\bar{D})$. If (62) and (63) are satisfied, (59) and (60) must hold for $\bar{D}$ and one can show that $C^*(\tilde{D}^-) < C^*(\bar{D})$ implies $\exists \ \underline{v} \ni C(D, \underline{v}) < C^*(\bar{D})$, a contradiction. Therefore, $C^*(\tilde{D}^-) > C^*(\bar{D})$ and (59) and (60) must hold. Q.E.D.

4. OPTIMAL CONTROL OF THE OFF-LINE SUBSYSTEMS

4.1. Results for General System Dynamics

As discussed in the general problem formulation, if the coordinator's control, $\underline{v}_i(t)$, for the ith subsystem switches from one to zero at t_1 and then switches back to one at $t_2 > t_1$, the local control for that subsystem is chosen to minimize $J_i(t_2 - t_1, u^i)$ given in (11) subject to $u^i \in U^i$ and the state differential equation and end conditions given in (12). It is reasonable to assume that during the off-line interval the subsystem could be maintained at the operating steady-state, since this steady-state is maintained for on-line operation. Hence, it is assumed $\exists \tilde{u}^i \in U^i \ni$

$$0 = f^i(\bar{x}^i, \tilde{u}^i, 0), \quad i = 1, \ldots, N. \tag{65}$$

The coordinator is then free to select any interval of time for a subsystem to be off-line since assumption (65) implies there is at lease one control for which the end conditions in (12) are satisfied. It will be shown in the following that for the particular models of interest, the minimizing control exists when it is known there exists at least one control satisfying the tpbv problem of (12).

Since the $f^i(x^i, u^i, 0)$ and $L_i(x^i, u^i)$ are independent of time for every $i=1, \ldots, N$, the properties of the optimal control problem for the off-line subsystems will be derived from the following generic model.

Consider a system described by the differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{66}$$

where $x(t) \in R^n$ is the system state, $u(t) \in R^m$ is the control variable, and $f : R^{n+m} \to R^n$ is continuous with continuous first partial derivatives WRT the components of x. Then for piecewise continuous u and a given $x(0)$, it is well-known there exists a unique continuous solution to (66). It is further assumed that for some compact set $U \subset R^m$, $u(t)$ must satisfy

$$u(t) \in U \quad \forall \quad t \in [0, T] \tag{67}$$

and that the solution to (66) must satisfy the boundary conditions

$$x(0) = \bar{x} \quad \text{and} \quad x(T) = \bar{x} \tag{68}$$

for a given $\bar{x} \in R^n$. The control is then selected to satisfy (66) through (68) while minimizing

$$J(T,u) = \int_o^T L(x,u)dt \tag{69}$$

where $L : R^{n+m} \to R^1$ is a non-negative function. Finally, assume $\exists\ \tilde{u} \in U$ such that

$$0 = f(\bar{x},\tilde{u}). \tag{70}$$

Now, if a piecewise continuous control exists which satisfied (66) through (68) and minimizes $J(T,u)$ in (69), let the minimal value be indicated by $J^*(T)$ given as

$$J^*(T) = \min_{u \in U} \int_o^T L(x,u)dt, \tag{71}$$

where the minimization is subject to the above constraints.

Of particular interest to the coordinator is how the cost of the off-line controls varies as a function of the off-line time interval. Therefore, in terms of the generic optimal control problem defined above, we wish to determine the characteristics of the function $J^*(T)$ given in (71). We first give some results concerning the special case when $J^*(\cdot)$ is monotone increasing and the state equation is of the general nature of (66). Then the linear-time invariant case of a particular "fuel optimal" problem is considered leading to formulas for the computation of $\frac{\partial J^*}{\partial T}$.

Lemma 14. If $J^*(T)$ given by (71) is monotone increasing for $T \in (T_1,T_2)$, then it is continuous for all $T \in (T_1,T_2)$, is differentiable almost everywhere on (T_1,T_2) and if the derivative exists at $\hat{T} \in (T_1,T_2)$,

$$0 \le \left.\frac{\partial J^*}{\partial T}\right|_{T=\hat{T}} \le L(\bar{x},\tilde{u}). \tag{72}$$

Proof. $J^*(T) \le J^*(T+\Delta) \le J^*(T) + \Delta \cdot L(\bar{x},\tilde{u})$, hence, taking the limit as $\Delta \to 0$

gives right continuity. Suppose $\lim_{\Delta\to 0^+} J^*(T-\Delta) = J^*(T^{(-)}) < J^*(T)$. Let $\alpha = J^*(T) - J^*(T^{(-)}) > 0$ and select $\tilde{\Delta} > 0$ such that $L(\bar{x},\tilde{u})\cdot\tilde{\Delta} < \alpha$. Then

$$J^*(T) \le J^*(T-\tilde{\Delta}) + \tilde{\Delta}\cdot L(\bar{x},\tilde{u}) < J^*(T^{(-)}) + \alpha = J^*(T),$$

which is a contradiction, proving left continuity. Differentiability a.e. follows from a theorem of real analysis([26],p. 76). Finally, for all $\Delta > 0$ it is clear that

$$0 \le \frac{J^*(T+\Delta) - J^*(T)}{\Delta} \le L(\bar{x},\tilde{u})$$

and

$$0 \le \frac{J^*(T-\Delta) - J^*(T)}{-\Delta} \le L(\bar{x},\tilde{u}),$$

proving (72).

The following two lemmas establish conditions under which $J^*(T)$ is in fact monotone increasing.

For $\tilde{x} \in R^n$, define $J_1^*(\tilde{x})$ and $J_2^*(\tilde{x},T)$

$$J_1^*(\tilde{x}) = \min_{u,T} \int_o^T L(x,u)dt \tag{73}$$

where the minimization is subject to (66), (67) and the end conditions

$$x(0) = \bar{x}, \quad x(T) = \tilde{x}, \tag{74}$$

and

$$J_2^*(\tilde{x},T) = \min_{u} \int_o^T L(x,u)dt \tag{75}$$

where the minimization is subject to (66), (67) and the end conditions

$$x(0) = \tilde{x}, \quad x(T) = \bar{x}. \tag{76}$$

<u>Lemma 15</u>. For $J^*(T)$ in (71), $J^*(T)$ is monotone increasing WRT T whereever it

exists if

$$J_1^*(\tilde{x}) + J_2^*(\tilde{x},T) \geq J^*(T) \tag{77}$$

for all $\tilde{x}$ and T such that $J_1^*(\tilde{x})$ and $J_2^*(\tilde{x},T)$ of (73) and (75), respectively, both exist.

Proof. Necessity: Suppose $J^*(T)\uparrow$ and $\exists\, T', \tilde{x}' \ni$

$$J_1^*(\tilde{x}') + J_2^*(\tilde{x}',T') < J^*(T')$$

Let $\tau^*(\tilde{x}')$ be the minimal minimizing time corresponding to the minimizing control and trajectory for $J_1^*(\tilde{x}')$. Then $\tau^*(\tilde{x}') > 0$ and, from the definition of $J^*(\cdot)$,

$$J^*[T' + \tau^*(\tilde{x}')] \leq J_1^*(\tilde{x}') + J_2^*(\tilde{x}',T'),$$

which implies $J^*[T' + \tau^*(\tilde{x}')] < J^*(T')$, contradicting the monotonicity of $J^*(\cdot)$.

Sufficiency: Suppose (77) holds but $\exists\; 0 < T_1 < T_2 \ni J^*(T_1) > J^*(T_2)$. Let $x^*(t,T_2)$, $u^*(t,T_2)$ be the optimal control and trajectory, respectively, corresponding to $J^*(T_2)$ and define $\delta = T_2 - T_1$. If $\tilde{x}^* \triangleq x^*(\delta,T_2)$, then the following relations hold:

$$J_2^*(\tilde{x}^*,T_1) = \int_\delta^{T_2} L[x^*(t,T_2),u^*(t,T_2)]\,dt \tag{78}$$

which follows from the principle of optimality and the time independence of $f(x,u)$ and $L(x,u)$;

$$J_1^*(\tilde{x}^*) \leq \int_0^\delta L[x^*(t,T_2),u^*(t,T_2)]\,dt \tag{79}$$

which follows from the definition of $J_1^*(\tilde{x})$.

Summing (78) and (79) gives

$$J_1^*(\tilde{x}^*) + J_2^*(\tilde{x}^*,T_1) \leq J^*(T_2) < J^*(T_1),$$

which contradicts (77), proving the lemma.

For the special case when the state is a scalar we have the general result:

Lemma 16. If $x \in R^1$, then $J^*(T)$ of (71) is monotone increasing.

Proof. By lemma 15, the lemma is false if $\exists\ \tilde{x}'$, $T \ni$

$$J_1^*(\tilde{x}) + J_2^*(\tilde{x},T) < J^*(T). \tag{80}$$

Assume such an $\tilde{x}$ and T exists and let $u_1(t)$, $x_1(t)$ denote the optimal control and state trajectory corresponding to $J_1^*(\tilde{x})$ and let T_1 be the corresponding minimal minimizing time. Let $u_2(t)$, $x_2(t)$ be the control and state trajectory corresponding to $J_2^*(\tilde{x},T)$. We show that (80) implies there exists a control satisfying (67) and (68) for which the cost is less than $J^*(T)$.

Case 1: $\exists\ \tilde{t} \in [0, \min(T,T_1)] \ni x_1(\tilde{t}) = x_2(\tilde{t})$.

Let a control, $\tilde{u}(t)$, be defined by

$$\hat{u}(t) = \begin{cases} u_1(t) & 0 \le t \le \tilde{t} \\ u_2(t) & \tilde{t} < t \le T. \end{cases}$$

Then the state trajectory generated by $\hat{u}(t)$ from (66) with $x(0) = \bar{x}$ is given by $\hat{x}(t)$ which satisfies:

$$\hat{x}(t) = \begin{cases} x_1(t) & 0 \le t \le \tilde{t} \\ x_2(t) & \tilde{t} < t \le T, \end{cases}$$

since $x_1(t)$ satisfies (66) for $u_1(t)$ and $x_2(t)$ satisfies (66) for $u_2(t)$. Now, $\hat{x}(0) = x_1(0) = \bar{x}$ and $x(T) = x_2(T) = \bar{x}$. Furthermore,

$$\int_0^T L(\hat{x},\hat{u})dt = \int_0^{\tilde{t}} L(x_1,u_1)dt + \int_{\tilde{t}}^T L(x_2,u_2)dt$$

$$\le J_1^*(\tilde{x}) + J_2^*(\tilde{x},T),$$

so by (80), $\int_o^T L(\hat{x},\hat{u})dt < J^*(T)$.

<u>Case 2</u>: $x_1(t)$ and $x_2(t)$ do not intersect.

First note that,

$$\forall\ t \in (0,T_1), \quad x_1(t) \notin \{\tilde{x},\bar{x}\}, \tag{81}$$

since, if for some such t $x_1(t) = \tilde{x}$ or $x_1(t) = \bar{x}$, there would be a time interval shorter than T_1 for which a control and state trajectory could satisfy (66), (67) and (74) with cost less than or equal to $J_1^*(\tilde{x})$, and this would contradict the definition of T_1.

Secondly, we note

$$T_1 < T. \tag{82}$$

Suppose (82) were note the case, i.e. $T_1 \geq T$. If $\tilde{x} > \bar{x}$, then $x_1(0) = \bar{x} < x_2(0) = \tilde{x}$ and by the continuity of $x_1(t)$ and $x_2(t)$ and the fact they do not intersect, $x_1(t) < x_2(t)\ \forall\ t \in [0,T]$. In particular, $x_1(T) < x_2(T) = \bar{x}$, which implies $\exists\ \tilde{t} \in [T,T_1) \ni x_1(\tilde{t}) = \bar{x}$, since $x_1(T_1) > \bar{x}_1$ which contradicts (81), if $T_1 > T$. If $T_1 = T$ we have $x_1(T) = \tilde{x} < \bar{x}$ contradicting the assumption. For $\tilde{x} < \bar{x}$ a similar contradiction is reached. Finally, if $\tilde{x} = \bar{x}$, then $T_1 = 0$ and $J_1^*(\hat{x}) = 0$ implying $J_2^*(\tilde{x},T) = J^*(T)$, contradicting (80). Therefore, (82) is true.

Thirdly, it must be the case that $\exists\ \tilde{t} \in [0, T - T_1] \ni$

$$x_2(\tilde{t}) = x_2(\tilde{t} + T_1). \tag{83}$$

To prove (83), assume $\tilde{x} > \bar{x}$ and suppose $x_2(t) \neq x_2(t + T_1)\ \forall t \in [0, T - T_1]$. Then, either $x_2(t) > x_2(t + T_1)$ or $x_2(t) < x_2(t + T_1)\ \forall$ such t, and since $x_2(0) = \tilde{x} > x_1(0) = \bar{x}$, by assumption, $x_2(t) > x_1(t)\ \forall\ t \in [0,T_1]$. In particular, $x_2(T_1) > x_1(T_1) = \tilde{x}$ and therefore, $x_2(0) = \tilde{x} < x_2(0 + T_1)$, which implies

$$x_2(t) < x_2(t+T_1) \ \forall\ t \in [0, T-T_1]. \tag{84}$$

Now, select $K \in \{1,2,\ldots\}$ such that $KT_1 < T \leq (K+1)T_1$ and from (84) we have:

$$x_2(T-KT_1) < x_2(T-(K-1)T_1) < \ldots < x_2(T-T_1) \leq x_2(T) = \bar{x}.$$

Since $0 < T - KT_1 \leq T_1$, we have $x_1(T-KT_1) < x_2(T-KT_1)$ which implies $x_1(T-KT_1) < \bar{x}$. But, $x_1(T_1) = \tilde{x} > \bar{x} \rightarrow \exists\ \hat{t} \in [T-KT_1, T_1) \ni x_1(\hat{t}) = \bar{x}$, which contradicts (81). A similar contradiction is obtained when $\tilde{x} < \bar{x}$, which proves (83).

Therefore, define the following control, $\tilde{u}(t)$:

$$\tilde{u}(t) \triangleq \begin{cases} u_1(t) & 0 \leq t \leq T_1 \\ u_2(t-T_1) & T_1 < t \leq \tilde{t} + T_1 \\ u_2(t) & \tilde{t} + T_1 < t \leq T, \end{cases}$$

where $\tilde{t}$ is a point satisfying (83). Then, for $x(0) = \bar{x}$, the trajectory satisfying (66) with control $\tilde{u}(t)$ is given by

$$\tilde{x}(t) \triangleq \begin{cases} x_1(t) & 0 \leq t \leq T_1 \\ x_2(t-T_1) & T_1 < t \leq \tilde{t} + T_1 \\ x_2(t) & \tilde{t} + T_1 < t \leq T. \end{cases}$$

Hence, $\tilde{x}(T) = \bar{x}$ and we have

$$\int_o^T L(\tilde{x},\tilde{u})dt = \int_8^{T_1} L(x_1,u_1)dt + \int_o^{\tilde{t}} L(x_2,u_2)dt + \int_{\tilde{t}+T_1}^{T} L(x_2,u_2)dt$$

$$\leq J_1^*(\tilde{x}) + J_2^*(\tilde{x},T) < J^*(T),$$

again contradicting the definition of $J^*(T)$, which proves the lemma.

Lemma 10 cannot be generalized to unrestricted systems of higher order so that the applicability of lemma 14 may be very restricted. Even for the very restricted class of optimal control problems to be considered below it is not true that $J^*(T)$ is necessarily monotone increasing which means other approaches are needed to analyze the properties of $J^*(T)$ and $\frac{\partial J^*}{\partial T}$.

4.2. Linear Time Invariant Systems

Consider the linear autonomous system state equation

$$\dot{x} = Ax + Bu \tag{85}$$

and let the control constraint set, U be given by

$$U = \{u \in R^m \mid 0 \leq u_i \leq \bar{u}_i,\ \bar{u}_i > 0,\ i = 1,\ldots,m\}. \tag{86}$$

For controls satisfying (67) with U given by (86) and solving the tpbv problem (85) and (68) for a given $\bar{x} \in R^n$, let the cost be given by

$$J(T,u) = \int_o^T k'u dt, \quad k \in R^m \tag{87}$$

where

$$k_i > 0, \quad i=1,\ldots,m, \quad k = [k_1,\ldots,k_m]'. \tag{88}$$

This is a type of "fuel optimal" problem, if the components of the control are considered rates of fuel consumption with unit costs given by the k_i, i=1,...,m. The properties of the optimal solution to this problem are considered in the sequel. For now, assume for $T = \hat{T}$ there exists an admissible control, $u^*(t,\hat{T})$, for $0 \leq t \leq \hat{T}$, and corresponding state trajectory, $x^*(t,\hat{T})$, satisfying (85) and the boundary constraints (68) for which $J[\hat{T},u^*(t,\hat{T})] = J^*(\hat{T})$, the minimal value of the cost (87).

It is well-known from the maximum principle that there exists a costate trajectory, $\lambda^*(t,\hat{T})$, $t \in [0,\hat{T}]$, which satisfies for some $\lambda_o^*(\hat{T}) \in R^n$,

$$\dot{\lambda}^*(t,\hat{T}) = -A'\lambda^*(t,\hat{T}) \tag{89}$$

with initial condition

$$\lambda^*(0,\hat{T}) = \lambda_o^*(\hat{T}) \tag{90}$$

such that for i=1,...,m,

$$u_i^*(t,\hat{T}) = \begin{cases} 0 & \text{if} \quad b_i'\lambda^*(t,\hat{T}) + k_i > 0 \\ \bar{u}_i & \text{if} \quad b_i'\lambda^*(t,\hat{T}) + k_i < 0 \end{cases} \tag{91}$$

where

$$B = [b_1 \vdots b_2 \vdots \dots \vdots b_m], \quad b_i \in R^m, \ i=1,\dots,m. \tag{92}$$

For now, assume (91) determines $u_i^*(t,\hat{T})$ uniquely a.e. on $[0,\hat{T}]$. Now, suppose an optimal control can be found for all T in some neighborhood of $\hat{T}$, with the optimal control, state trajectory, costate trajectory and costate initial condition denoted, for each T, by $u^*(t,T)$, $x^*(t,T)$, $\lambda^*(t,T)$, $\lambda_o^*(T)$ respectively. Hence, $J^*(T)$ is defined in a neighborhood of T and formally from (87) we have

$$\frac{\partial J^*(T)}{\partial T}\bigg|_{T=\hat{T}} = k'u^*(\hat{T},\hat{T}) + \int_o^T k' \frac{\partial u^*(t,T)}{\partial T}\bigg|_{T=\hat{T}} dt, \tag{93}$$

where (93) is an application of Leibniz' rule. In the following an explicit form for (93) is derived and sufficient conditions are given for its validity.

Towards that end, make the following definitions. For $\lambda_o \in R^n$ let $\tilde{\lambda}(t,\lambda_o)$ be the solution to

$$\dot{\tilde{\lambda}}(t,\lambda_o) = -A'\tilde{\lambda}(t,\lambda_o), \quad t \geq 0, \tag{94}$$

with $\tilde{\lambda}(0,\lambda_o) = \lambda_o$.

Define for $t \geq 0$, $i=1,\ldots,m$, $f_i(t,\lambda_o)$ as

$$f_i(t,\lambda_o) = b_i' \tilde{\lambda}(t,\lambda_o) + k_i \tag{95}$$

and let $u_i(t,\lambda_o)$ be given by

$$\tilde{u}_i(t,\lambda_o) = \begin{cases} 0 & \text{if} \quad f_i(t,\lambda_o) \geq 0 \\ \bar{u}_i & \text{if} \quad f_i(t,\lambda_o) < 0 \end{cases}, \quad i=1,\ldots,m. \tag{96}$$

Let $\tilde{u}(t,\lambda_o) = [\tilde{u}_1(t,\lambda_o),\ldots,\tilde{u}_m(t,\lambda_o)]'$. Finally, define $\tilde{x}(t,\lambda_o)$ as the solution for $t \geq 0$ of

$$\dot{\tilde{x}}(t,\lambda_o) = A\,\tilde{x}(t,\lambda_o) + B\,\tilde{u}(t,\lambda_o) \tag{97}$$

with $\tilde{x}(0,\lambda_o) = \bar{x}$.

Now, define $F : R^{n+1} \to R^n$ by

$$F(T,\lambda_o) = \tilde{x}(T,\lambda_o) - \bar{x} \tag{98}$$

and note that for $T = \hat{T}$ and $\lambda_o = \lambda_o^*(\hat{T})$ as in (90) we have

$$F[\hat{T},\lambda_o^*(\hat{T})] = 0. \tag{99}$$

The general conditions for which it is guaranteed $\lambda_o^*(T)$ is a continuous function of T in a neighborhood of $\hat{T}$ and a formula for its derivative are to be derived using the following implicit Function theorem.

Theorem 4. Let $F_{\lambda_o}[T,\lambda_o]$ denote the derivative of $F[T,\lambda_o]$ WRT λ_o evaluated at (T,λ_o) and $F_T(T,\lambda_o)$ denotes the derivative WRT T. Then, if for a given $(\hat{T},\hat{\lambda}_o)$ we have,

$$F(\hat{T},\hat{\lambda}_o) = 0, \tag{100}$$

$$F(T,\lambda_o) \text{ is continuous in a neighborhood of } (\hat{T},\hat{\lambda}_o), \tag{101}$$

$F_{\lambda_o}(T,\lambda_o)$, $F_T(T,\lambda_o)$ are both bounded and continuous in a neighborhood of $(\hat{T},\hat{\lambda}_o)$ (102)

$F(T,\lambda_o)$ is differentiable in a neighborhood of $(\hat{T},\hat{\lambda}_o)$, (103)

and $[F_{\lambda_o}(\hat{T},\hat{\lambda}_o)]^{-1}$ exists, (104)

then

$$\exists\, G : R^1 \to R^n \ni$$

G is continuous in a neighborhood of $\hat{\mathbf{T}}$, denoted by $\mathcal{N}(\hat{T})$, (105)

$F[T,G(T)] = 0$ for all T in $\mathcal{N}(\hat{T})$, (106)

$\hat{\lambda} = G(\hat{T})$ and (107)

$$\left.\frac{dG}{dT}\right|_{T=\hat{T}} = -[F_{\lambda_o}(\hat{T},\hat{\lambda}_o)]^{-1} F_T(\hat{T},\hat{\lambda}_o). \quad (108)$$

Proof. Pages 194-198 of [13].

To apply theorem 4 we first analyze some properties of the optimal control problem being considered and then state specific conditions under which the optimal solution leads to a satisfaction of the hypothesis of theorem 4. It is also shown that the G(T) in (105) actually equals $\lambda_o^*(T)$ in the entire neighborhood, $\mathcal{N}(\hat{T})$.

Consider again the equations (68) and (85) through (88) defining the optimal control problem. The following results characterize the optimal solution and most of the proofs will be omitted since the arguments are similar to known results concerning a somewhat different fuel-optimal control problem in [14]. We first make the following standard definition:

Definition: If for an optimal solution to (85)-(88) which satisfies (89)-(91) there is for some $i \in \{1,\ldots,m\}$ an interval in $[0,T]$ such that $b_i'\lambda^*(t,T) + k_i = 0$

on the entire interval, then the problem is called singular.

Since we wish for (91) to uniquely determine the optimal control almost everywhere, it is desirable to be assured the problem is not singular. The following are sufficient conditions for this to be the case.

Lemma 17. If A is nonsingular and the pairs (A,b_i) are controllable for all $i=1,\ldots,m$, then the optimal control problem (85)-(88) is not singular for any T.

When A and B of (85) satisfy the hypothesis of lemma 17 the problem will be called "normal" (note the usual definition of a normal problem does not include the nonsingularity of A, however, for convenience in the following it will be included). The following lemmas deal with the existence and uniqueness of an optimal control for a normal problem of the type in (85)-(88).

Lemma 18. An optimal control minimizing $J(T,u)$ (87) exists if there exists at least one control in the constraint set of (86) for each $t\in[0,T]$, such that the tpbv problem, (68) and (85) is satisfied.

Proof. pp. 127 ff. in [15].

From the general assumption (70), lemma 18 implies that for any time interval an optimal control exists. Note, lemma 18, did not require a normal problem.

Lemma 19. If the problem (85)-(88) is normal and an optimal control exists, then it is unique.

Proof. Similar to proof of Theorem 6-14 in [14].

Note that by uniqueness one means up to a set of measure zero since changing a control at isolated points affects neither the solution to (85) nor the value of the integral (87).

Lemma 20. For a normal problem given by (85)-(88) if a control exists with corresponding state and costate trajectories satisfying (68) and (90)-(91) for some costate initial condition, then the control is unique in that for no other control which solves the tpbv problem (85) and (68) can (89)-(91) be satisfied for any costate initial condition.

Proof. Similar to proof of Theorem 6-15 in [14].

We now apply these results to an analysis of $F(T,\lambda_o)$ and the condition (99).

Lemma 21. For a normal control problem given by (85)-(88), if for $\hat{T} > 0$ there exists a $\hat{\lambda}_o \in R^n$ such that $F(\hat{T},\hat{\lambda}_o) = 0$, where $F(T,\lambda_o)$ is defined by (94) to (98), then $\tilde{u}(t,\hat{\lambda}_o)$ of (96) is the unique optimal control, for $T=\hat{T}$.

Proof. From the definition of $F(T,\lambda_o)$ and equations (94) to (98) it is clear that if $F(\hat{T},\hat{\lambda}_o) = 0$, then $\tilde{\lambda}(t,\hat{\lambda}_o)$, $\tilde{u}_i(t,\hat{\lambda}_o)$ and $\tilde{x}(t,\hat{\lambda}_o)$ satisfy the necessary conditions (89)-(91) for an optimal control as well as the tpbv problem (85) and (68), for $T = \hat{T}$. Hence, by lemmas 19 and 20 $\tilde{u}(t,\hat{\lambda}_o)$ is the unique optimal control for $T = \hat{T}$.

Lemma 22. $F(T,\lambda_o)$ is continuous for all $\lambda_o \in R^n$, $T \geq 0$.

Proof. This follows from the definition of $F(T,\lambda_o)$ and the continuity of the solution of a system of differential equations with respect to the initial conditions.

We now evaluate $F_{\lambda_o}[T,\lambda_o] = \dfrac{\partial \tilde{x}(T,\lambda_o)}{\partial \lambda_o}$, following the work of Stefenak [16] on the sensitivity of solutions of differential equations with discontinuous elements. Define the matrices $\Lambda(t,\lambda_o)$ and $\underline{X}(t,\lambda_o)$ as

$$\underline{X}(t,\lambda_o) \triangleq \frac{\partial \tilde{x}(t,\lambda_o)}{\partial \lambda_o} \tag{109}$$

and

$$\Lambda(t,\lambda_o) \triangleq \frac{\partial\tilde{\lambda}(t,\lambda_o)}{\partial\lambda_o} \tag{110}$$

where $\tilde{\lambda}(t,\lambda_o)$ and $\tilde{x}(t,\lambda_o)$ are defined by (94) and (97), respectively, Then, from (94), $\Lambda(t,\lambda_o)$ is the solution to the matrix differential equation

$$\dot{\Lambda}(t,\lambda_o) = -A''\Lambda(t,\lambda_o) \tag{111}$$

with

$$\Lambda(0,\lambda_o) = I.$$

The differential equation for $\tilde{x}(t,\lambda_o)$ involves piecewise constant control values. Consider the set of switch points for $\tilde{u}(t,\lambda_o)$ on $[0,\infty)$, $\mathcal{T}(\lambda_o)$ defined by

$$\mathcal{T}(\lambda_o) = \{t\in[0,\infty) \mid \tilde{u}_i(t^-,\lambda_o) \neq \tilde{u}_i(t^+,\lambda_o) \text{ for some } i\in\{1,\ldots,m\}\}. \tag{112}$$

For a normal control problem, $\mathcal{T}(\lambda_o)$ is a set of isolated points on $[0,\infty)$. Define $N(T,\lambda_o)$ as the number of points in $\mathcal{T}(\lambda_o)$ less than or equal to T. Then we can define

$$\{t_\nu\}_{\nu=1}^{N(T,\lambda_o)} = \mathcal{T}(\lambda_o) \cap [0,T] \tag{113}$$

where

$$0 \le t_1 < \ldots < t_{N(T,\lambda_o)} \le T. \tag{114}$$

Define $\Upsilon : R^{n+1} \to \mathcal{P}[\{1,\ldots,m\}]$ as

$$\Upsilon(t,\lambda_o) = \{i\in\{1,\ldots,m\} \mid \tilde{u}_i(t^-,\lambda_o) \neq \tilde{u}_i(t^+,\lambda_o)\}, \tag{115}$$

indicating the components of $\tilde{u}(t,\lambda_o)$ which switch at t. Note that if $t \notin \mathcal{T}(\lambda_o)$ then $\Upsilon(t,\lambda_o) = \phi$. Now, the differential equation satisfied by $X(t,\lambda_o)$ between switch points of $\tilde{u}(t,\lambda_o)$ is

$$\dot{\bar{X}}(t,\lambda_o) = A\,\bar{X}(t,\lambda_o). \tag{116}$$

Clearly, since $\tilde{x}(0,\lambda_o) = \bar{x}$ for all $\lambda_o \in R^n$,

$$\bar{X}(0,\lambda_o) = 0_{nxn}. \tag{117}$$

At each point $t_\nu \in \mathcal{J}(\lambda_o)$ there is a jump in $\bar{X}(t,\lambda_o)$ which is given by

$$\bar{X}(t_\nu^+,\lambda_o) - \bar{X}(t_\nu^-,\lambda_o) = \sum_{i\in\Upsilon(t_\nu,\lambda_o)} b_i[\tilde{u}_i(t_\nu^+,\lambda_o) - \tilde{u}_i(t_\nu^-,\lambda_o)]\left[\frac{\partial t_\nu^i}{\partial\lambda_o}\right]', \tag{118}$$

where $\frac{\partial t^i}{\partial\lambda_o}$ is the sensitivity of the switch time for the ith component of the control. This is computed by applying the implicit function theorem (theorem 4) to the switching functions $f_i(t,\lambda_o)$, $i=1,\ldots,m$, of (95).

Lemma 23. For the switch time $t_\nu \in \mathcal{J}(T,\lambda_o)$, and $i \in \Upsilon(t_\nu,\lambda_o)$ if

$$b_i' A' \tilde{\lambda}(t_\nu,\lambda_o) \neq 0 \tag{119}$$

then, the switch time sensitivity of the ith component of the control with respect to λ_o, $\frac{\partial t_\nu^i}{\partial\lambda_o}$, is given by

$$\frac{\partial t_\nu^i}{\partial\lambda_o} = \frac{\Lambda'(t_\nu,\lambda_o)b_i}{b_i' A' \tilde{\lambda}(t_\nu,\lambda_o)}. \tag{120}$$

Proof. From (95) and the definitions of t_ν, and $\Upsilon(t_\nu,\lambda_o)$, $f_i(t_\nu,\lambda_o) = b_i'\lambda(t,\lambda_o) + k_i = 0$. Clearly, $f_i(t,\lambda_o)$ is continuous and differentiable. $\frac{\partial f_i}{\partial t} = -b_i' A' \tilde{\lambda}(t,\lambda_o)$ is continuous, bounded and invertible by (119). Furthermore, $\frac{\partial f_i}{\partial\lambda_o} = \Lambda'(t,\lambda_o)b_i$ is also continuous and bounded. Therefore, theorem 4 implies (120).

Combining (116), (117), (118) and lemma 23 gives,

Lemma 24. The sensitivity of $\tilde{x}(t,\lambda_o)$ with respect to λ_o is given by $\bar{X}(t,\lambda_o)$ by (121) and (122) below for all $t \notin \mathcal{J}(\lambda_o)$, if for all $t_\nu \in \mathcal{J}(\lambda_o)$, $b_i' A' \tilde{\lambda}(t_\nu,\lambda_o) \neq 0$ when $i \in \Upsilon(t_\nu,\lambda_o)$.

$$X(t,\lambda_o) = \sum_{\nu=1}^{N(t,\lambda_o)} e^{A(t-t_\nu)} [X(t_\nu^+,\lambda_o) - X(t_\nu^-,\lambda_o)] \tag{121}$$

where

$$X(t_\nu^+,\lambda_o) - X(t_\nu^-,\lambda_o) = \sum_{i\in\gamma(t_\nu,\lambda_o)} \frac{\tilde{u}_i(t_\nu^+,\lambda_o) - \tilde{u}_i(t_\nu^-,\lambda_o)}{b_i' A' \tilde{\lambda}(t_\nu,\lambda_o)} b_i b_i' \Lambda(t_\nu,\lambda_o). \tag{122}$$

Proof. (122) follows from (118) and lemma 23, and (121) is the result of integrating (116) subject to (117) and (118).

Conditions for which theorem 4 may be applied to the fuel optimal control problem can now be stated.

Theorem 5. If the control problem (85) to (88) is normal and for $\hat{T} > 0$, $\lambda_o^*(\hat{T})$ is the initial condition for a costate trajectory corresponding to an optimal control and optimal state trajectory satisfying (68) and (89) to (91), and, provided the following conditions are satisfied;

$$\hat{T} \notin \mathcal{J}[\lambda_o^*(\hat{T})], \tag{123}$$

for all $t_\nu \in \mathcal{J}[\lambda_o^*(\hat{T})] \cap [0,T]$ and $i \in \gamma[t_\nu,\lambda_o^*(\hat{T})]$,

$$b_i' A' \tilde{\lambda}[t_\nu,\lambda_o^*(\hat{T})] \neq 0 \tag{124}$$

and for $X[t,\lambda_o^*(\hat{T})]$ given by (121)

$$X[\hat{T},\lambda_o^*(\hat{T})] \text{ is non-singular,} \tag{125}$$

then, there exists a continuous function $\lambda_o^*(T)$ defined in a neighborhood of $\hat{T}$, $\mathcal{N}(\hat{T})$, such that the control $\tilde{u}[t,\lambda_o^*(T)]$ generated through equations (94)-(96) is the unique optimal control for $T \in \mathcal{N}(\hat{T})$ and $\frac{\partial\lambda_o^*(T)}{\partial T}$ evaluated at $\hat{T}$ is given by

$$\frac{\partial\lambda_o^*}{\partial T}\bigg|_{T=\hat{T}} = -\{X[\hat{T},\lambda_o^*(\hat{T})]\}^{-1} \cdot \{A\bar{x} + B\tilde{u}[\hat{T},\lambda_o^*(\hat{T})]\} \tag{126}$$

Proof. We show that for $F[\hat{T},\lambda_o^*(\hat{T})]$, the hypothesis (100)-(104) of theorem 4 are satisfied. Clearly $F[\hat{T},\lambda_o^*(\hat{T})] = 0$ since $X^*[\hat{T},\lambda_o^*(\hat{T})] = \bar{x}$. (101) holds by lemma 22. From the definitions of $F(T,\lambda_o)$ and $X(t,\lambda_o)$ we have formally,

$$F_{\lambda_o}[T,\lambda_o] = X[T,\lambda_o], \tag{127}$$

which, by assumptions (123) and (124) and lemma 24, exists, is continuous and bounded in some neighborhood of $(\hat{T},\lambda_o^*(\hat{T}))$. Furthermore, whenever $T \notin \mathcal{J}(\lambda_o)$, from (98),

$$F_T[T,\lambda_o] = A\tilde{x}(T,\lambda_o) + B\tilde{u}(T,\lambda_o). \tag{128}$$

Hence, by assumption (123), $F_T[T,\lambda_o]$ is continuous and bounded in some neighborhood of $(\hat{T},\lambda_o^*(\hat{T}))$. Therefore, (102) is satisfied. Since $F_{\lambda_o}(T,\lambda_o)$ and $F_T(T,\lambda_o)$ exist and are continuous in a neighborhood of $(\hat{T},\hat{\lambda}_o)$, $F(T,\lambda_o)$ is differentiable in that neighborhood, satisfying (103). Finally, assumption (125) and (127) imply $\{F_{\lambda_o}[\hat{T},\lambda_o^*(\hat{T})]\}^{-1}$ exists.

Theorem 4 then implies the existence of a function $G(T)$ satisfying (105)-(108). It remains to be shown that $\lambda_o^*(T) = G(T)$, i.e. if $G(T)$ is used as an initial condition, the resulting control, $\tilde{u}[t,G(T)]$ is in fact optimal for $T \in \mathcal{N}(\hat{T})$. This is in fact the case since $F[T,G(T)] = 0$, by lemma 21. (126) is (108) using (127) and (128). Q.E.D.

Now, returning to the properties of $J^*(T)$ and $\frac{\partial J^*}{\partial T}$ in (93), consider the integrand $k' \frac{\partial u^*(t,T)}{\partial T}$. If $\lambda_o^*(T)$ generates an optimal control for each T through equations (94) to (96), then

$$u_i^*(t,T) = \tilde{u}_i[t,\lambda_o^*(T)] \quad \forall\, t \notin \mathcal{J}[\lambda_o^*(T)]. \tag{129}$$

Formally, we have

$$\frac{\partial u_i^*(t,T)}{\partial T}\bigg|_{T=\hat{T}} = \lim_{\Delta\to 0} \frac{\tilde{u}_i[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}_i[t,\lambda_o^*(\hat{T})]}{\Delta} \tag{130}$$

which, by the piecewise constant nature of $\tilde{u}_i(t,\lambda_o)$, is zero for all $t \notin \mathcal{J}[\lambda_o^*(\hat{T})]$ if $\lambda_o^*(T)$ is continuous in a neighborhood of $\hat{T}$. Consider (130) at a switch point $t_\nu \in \mathcal{J}[\lambda_o^*(\hat{T})] \cap (0,\hat{T})$ and suppose $i \in \gamma[t_\nu,\lambda_o^*(\hat{T})]$. Considering $t_\nu = t_\nu^i(\lambda_o)$, as a function of the costate initial condition, under assumption (119), the sensitivity of $t_\nu^i(\lambda_o)$ with respect to λ_o is given by (120). Hence, provided

$$b_i' A' \tilde{\lambda}[t_\nu,\lambda_o^*(\hat{T})] \neq 0, \tag{131}$$

the variation of the switch time of the ith component of the control with respect to T at $T = \hat{T}$, may be written as

$$\frac{\partial t_\nu^i}{\partial T} = \left\langle \frac{\partial t_\nu^i}{\partial \lambda_o}\Bigg|_{\lambda_o=\lambda_o^*(\hat{T})}, \frac{\partial \lambda_o^*(T)}{\partial T}\Bigg|_{T=\hat{T}} \right\rangle \tag{132}$$

with $\frac{\partial t_\nu^i}{\partial \lambda_o}$ given by (120) and, provided the assumptions of theorem 5 hold, $\frac{\partial \lambda_o^*(T)}{\partial T}\Big|_{T=\hat{T}}$ given by (126)

The evaluation of the limit in (141) leads to delta-functions at each $t_\nu \in \mathcal{J}[\lambda_o^*(\hat{T})]$, i.e., in a neighborhood of $t_\nu \in \mathcal{J}[\lambda_o^*(\hat{T})] \cap (0,\hat{T})$, $\mathcal{N}(t_\nu)$, we have

$$\frac{\partial u_i(t,T)}{\partial T}\Bigg|_{T=\hat{T}} = \delta(t-t_\nu) \cdot \frac{\partial t_\nu}{\partial T}\Bigg|_{T=\hat{T}} \quad \text{for } t \in \mathcal{N}(t_\nu), \tag{133}$$

where $\frac{\partial t_\nu}{\partial T}$ is given by (132). The full evaluation of $\frac{\partial u^*(t,T)}{\partial T}$ may be expressed in terms of δ-functions and then substituted into (93) to evaluate $\frac{\partial J^*}{\partial T}$. Applying Leibniz' rule in such a case with generalized functions is, in fact, valid and the following result could be proved in that manner (see p. 77, [17]). However, the following theorem will be proved directly by considering the limit of the entire integral.

Theorem 6. Given a normal optimal control problem with state equation (85), control constraint (86), boundary conditions (68) and cost, (87), let $J^*(T)$ be the minimal value for the cost for time interval $[0,T]$. If for $\hat{T} > 0$, $\exists$ an initial costate $\hat{\lambda}_o$ such that the control $\tilde{u}(t,\hat{\lambda}_o)$ generated by (94)-(96) is optimal for $T = \hat{T}$ and if

$$\hat{T} \notin \mathcal{T}(\hat{\lambda}_o), \tag{134}$$

and for all $t_\nu \in \mathcal{T}(\hat{\lambda}_o) \cap [0,\hat{T}]$ with $i \in \Upsilon(t_\nu,\hat{\lambda}_o)$,

$$b_i' A' \tilde{\lambda}(t_\nu,\hat{\lambda}_o) \neq 0 \tag{135}$$

and

$$\underline{\overline{X}}(\hat{T},\hat{\lambda}_o) \text{ is non-singular,} \tag{136}$$

then, $J^*(T)$ is continuous and differentiable in a neighborhood of $\hat{T}$ with $\left.\frac{\partial J^*}{\partial T}\right|_{T=\hat{T}}$ given by

$$\left.\frac{\partial J^*(T)}{\partial T}\right|_{T=\hat{T}} = k'\tilde{u}(\hat{T},\hat{\lambda}_o) + \{[\underline{\overline{X}}(\hat{T},\hat{\lambda}_o)]^{-1}[A\bar{x} + B\tilde{u}(\hat{T},\hat{\lambda}_o)]\}' \times \left\{ \sum_{i=1}^{m} \sum_{j=1}^{N_i(\hat{T},\hat{\lambda}_o)} \left[\frac{\bar{u}_i k_i (-1)^j \xi_i(\hat{\lambda}_o)}{b_i' A' e^{-A' t_j^i} \hat{\lambda}_o} \right] e^{-A t_j^i} b_i \right\} \tag{137}$$

where $\underline{\overline{X}}(t,\lambda_o)$ is defined in (109),

$$\xi_i(\lambda_o) = \begin{cases} -1 & \text{if} \quad f_i(0,\lambda_o) \geq 0 \\ 1 & \text{if} \quad f_i(0,\lambda_o) < 0 \end{cases} \tag{138}$$

$$N_i(T,\lambda_o) = \# \text{ of } t_\nu \in \mathcal{T}\{\lambda_o\} \cap [0,T] \ni \Upsilon(t_\nu,\lambda_o) = i \tag{139}$$

and

$$\{t_j^i\}_{j=1}^{N_i(T,\lambda_o)} = \{t_i \in \mathcal{T}(\lambda_o) \cap [0,T] \mid \gamma(t_\nu,\hat{\lambda}_o) = i\} \tag{140}$$

with $0 \le t_1^i < \ldots < t^i_{N_i(T,\lambda_o)} \le T$ for $i=1,\ldots,m$.

Proof. If the derivative exists,

$$\left.\frac{\partial J^*}{\partial T}\right|_{T=\hat{T}} = \lim_{\Delta\to 0} \frac{J^*(\hat{T}+\Delta) - J^*(\hat{T})}{\Delta} = \lim_{\Delta\to 0} \frac{1}{\Delta}\{\int_o^{\hat{T}+\Delta} k'u^*(t,\hat{T}+\Delta)dt - \int_o^{\hat{T}} k'u^*(t,\hat{T})dt\}$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta}\{\int_o^{\hat{T}+\Delta} k'\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)]dt - \int_o^{\hat{T}} k'\tilde{u}[t,\lambda_o^*(\hat{T})]dt\}$$

where $\lambda_o^*(\hat{T}) = \hat{\lambda}_o$, and $\lambda_o^*(T)$ is the continuous, differentiable function in a neighborhood of $\hat{T}$ which generates through (94) to (96) the unique optimal control, $u^*(t,T) = \tilde{u}[t,\lambda_o^*(T)]$ for the time T. $\lambda_o^*(T)$ is guaranteed to exist by Theorem 5. Now, for $\Delta > 0$,

$$\frac{J^*(\hat{T}+\Delta) - J^*(\hat{T})}{\Delta} = \frac{1}{\Delta}\int_{\hat{T}}^{\hat{T}+\Delta} k'\tilde{u}[t, {}_o^*(\hat{T}+\Delta)]dt + \frac{1}{\Delta}\int_o^{\hat{T}} k'\{\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}[t,\lambda_o^*(\hat{T})]\}dt. \tag{141}$$

Since $\hat{T} \notin \mathcal{T}(\hat{\lambda}_o)$, the first term clearly gives

$$\lim_{\Delta\to 0^+} \int_{\hat{T}}^{\hat{T}+\Delta} k'\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)]dt = k'\tilde{u}[\hat{T},\hat{\lambda}_o] \tag{142}$$

because $\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)] \equiv \tilde{u}[\hat{T},\lambda_o^*(\hat{T})]$ in a neighborhood of $\hat{T}$ by the continuity of the switch functions (95) and the continuity of $\lambda^*(T)$.

Now, the continuity of the switch times WRT T further implies that given $\varepsilon > 0$, $\exists\ \Delta_\varepsilon > 0 \ni \forall\ 0 < \Delta \leq \Delta_\varepsilon$,

$$N(\hat{T},\hat{\lambda}_o) = N[\hat{T},\lambda_o^*(\hat{T}+\Delta)], \tag{143}$$

and

$$t_\nu[\lambda_o^*(\hat{T}+\Delta)] \in (t_\nu(\hat{\lambda}_o) - \varepsilon,\ t_\nu(\hat{\lambda}_o) + \varepsilon) \tag{144}$$

For a normal problem, the number of switch times in a finite interval is finite (this is proved in a later lemma) so that ε may be chosen small enough that

$$(t_{\nu_1}(\hat{\lambda}_o) - \varepsilon,\ t_{\nu_1}(\hat{\lambda}) + \varepsilon) \cap (t_{\nu_2}(\hat{\lambda}_o) - \varepsilon,\ t_{\nu}(\hat{\lambda}_o) + \varepsilon) = \phi \qquad (145)$$

$$\forall\ \nu_1,\ \nu_2 \in \{1,\ldots,N(\hat{T},\hat{\lambda}_o)\},\ \nu_1 \neq \nu_2.$$

Therefore, for $0 < \Delta \leqslant \Delta_{\varepsilon}$, the second term in (141) is,

$$\frac{1}{\Delta}\int_o^{\hat{T}} k'\{\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}[t,\lambda_o^*(\hat{T})]\}dt = \frac{1}{\Delta}\sum_{\nu=1}^{N(\hat{T},\hat{\lambda}_o)} \int_{\underline{t}_\nu}^{\overline{t}_\nu} k'\{\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}[t,\lambda_o^*(\hat{T})]\}dt, \qquad (146)$$

where

$$\underline{t}_\nu = \max\{t_\nu - \in,\ 0\} \text{ and } \overline{t}_\nu = \min\{t_\nu - \in,\ \hat{T}\}.$$

Now, suppose $\tilde{u}_i(t,\hat{\lambda}_o)$ switches from 0 to $\overline{u}_i$ at $t_\nu(\hat{\lambda}_o)$. Then

$$\int_{\underline{t}_\nu}^{\overline{t}_\nu} k_i\{\tilde{u}_i[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}_i[t,\lambda_o^*(\hat{T})]\}dt = -k_i\overline{u}_i[t_\nu[\lambda_o^*(\hat{T}+\Delta)] - t_\nu(\hat{\lambda}_o)]$$

and similarly if $\tilde{u}_i(t,\hat{\lambda}_o)$ switches from $\overline{u}_i$ to 0,

$$\int_{\underline{t}_\nu}^{\overline{t}_\nu} k_i\{\tilde{u}_i[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}_i[t,\lambda_o^*(\hat{T})]\}dt = k_i\overline{u}_i[t_\nu[\lambda_o^*(\hat{T}+\Delta)] - t_\nu(\hat{\lambda}_o)].$$

Since $\tilde{u}_i(t,\hat{\lambda}_o)$ switches between 0 and $\overline{u}_i$ according to (96), if we consider only the $t_\nu \in \mathcal{T}(\hat{\lambda}_o)$ where the ith component of the control switches, then, using definitions (138) to (140) we have

$$\sum_{j=1}^{N_i(\hat{T},\hat{\lambda}_o)} \int_{\underline{t}_j^i}^{\overline{t}_j^i} k_i\{\tilde{u}_i[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}_i[t,\lambda_o^*(\hat{T})]\}dt$$
$$= -\xi_i(\hat{\lambda}_o)\overline{u}_i k_i \sum_{j=1}^{N_i(\hat{T},\hat{\lambda}_o)} \{t_j^i[\lambda_o^*(\hat{T}+\Delta)] - t_j^i[\lambda_o^*(\hat{T})]\}. \qquad (147)$$

Dividing (147) by Δ, taking the limit as $\Delta \to 0^+$, and summing over all i=1,...,m gives,

$$\lim_{\Delta\to 0^+}\frac{1}{\Delta}\int_o^{\hat{T}} k'\{\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}[t,\lambda_o^*(\hat{T})]\}dt = -\sum_{i=1}^{m}\sum_{j=1}^{N_i(\hat{T},\hat{\lambda}_o)} \xi_i(\hat{\lambda}_o)\overline{u}_i k_i \left\langle \frac{\partial t_\nu^i}{\partial\lambda_o}\Big|_{\lambda_o=\hat{\lambda}_o}, \frac{\partial\lambda_o^*}{\partial T}\Big|_{T=\hat{T}} \right\rangle.$$

From (120), (126) and the fact that the solution, $\Lambda(t,\lambda_o)$, to (111) is given by $e^{-A't}$ and $\tilde{\lambda}(t,\lambda_o) = e^{-A't}\lambda_o$, from (94),

$$\left\langle \frac{\partial t_\nu^i}{\partial \lambda_o}\bigg|_{\lambda_o=\hat{\lambda}_o}, \frac{\partial \lambda_o^*(T)}{\partial T}\bigg|_{T=\hat{T}} \right\rangle = -\{[X(\hat{T},\hat{\lambda}_o)]^{-1}\cdot[A\overline{x}+B\tilde{u}(\hat{T},\hat{\lambda}_o)]\}'$$

$$\cdot\{\frac{1}{b_i'A'e^{-A't}\hat{\lambda}_o} e^{-At}b_i\}.$$

Combining the last two equations gives the second term of (137).

For $\Delta \to 0^-$ the above analysis applies except that for $\Delta < 0$ we write

$$\frac{J^*(\hat{T}+\Delta) - J^*(\hat{T})}{\Delta} = \frac{1}{\Delta}\int_o^{\hat{T}} k'\{\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)] - \tilde{u}[t,\lambda_o^*(\hat{T})]\}dt$$

$$- \frac{1}{\Delta}\int_{\hat{T}+\Delta}^{\hat{T}} k'\tilde{u}[t,\lambda_o^*(\hat{T}+\Delta)]dt. \quad (148)$$

The last term gives the same limit as in (142) and clearly the first term approaches the second term of (137) using the same argument applied to (146). Hence, the left and right derivatives of $J^*(T)$ exist and are equal, proving the theorem. Q.E.D.

The coordinator may use the result of theorem 6 to compute the variation of the off-line costs WRT variations in the time intervals dictated by $\underline{v}(t)$. Since this computation depends upon knowledge of the optimal control for a given time interval, methods for solving the optimal control problem must be available. One such method is that of Stefenale [16] which employs a gradient search or Newton's method to find the optimal initial costate.

5. APPLICATION TO THE UNIT COMMITMENT PROBLEM

5.1. Application of the General Results

The generic problem description of Section 1.2 contains most of the salient features of the thermal unit commitment problem for power systems as it is normally formulated. The model developed in Section 1.2 has the added feature of providing for a more precise modeling of the off-line control, as discussed in Section 1.1. The primary element of the unit commitment problem not included in this formulation is the stochastic nature of the prediction of demand and predicted machine availability. Although these are important considerations, they were not taken into account in this initial inquiry since even the deterministic problem demands further research and currently such stochastic aspects are disregarded or included only in a very heuristic manner in practical optimization schemes.

By separating the problem into the two natural divisions of a steady-state optimization (Section 3) and several dynamic optimal control problems (Section 4), the relation of the general formulation to the usual approach to unit commitment is evident. The pointwise optimal coordination of the on-line subsystems corresponds to what is known as economic dispatch for the power system application. Economic dispatch is well developed theoretically [18] and is almost universally used in some form throughout the electric utility industry [19]. On the other hand, the formulation and modeling of the shut-down and start-up of thermal generating units as an optimal control problem is done in a very rough manner in the literature and requires considerable research [20]. The computation of the off-line costs is currently done using an empirical determination of the parameters for a simple start-up cost vs. down-time formula representing an exponential cooling rate or some computationally advantageous approximation there of [3,21]. This modeling problem is discussed further in Section 5.2.

Once a reasonable model is available, the results of the previous sections may be employed to enhance and extend current unit commitment optimization schemes. One possible application of the results of Section 3 would be to use the properties of the optimal cost functions given there to enhance the dynamic and integer programming algorithms which are currently used to find sub-optimal commitment policies. This is mentioned briefly in Section 5.3 as a area for further research.

Another application of the results, particularly theorems 1 and 6, would be to compute the gradient of the cost for a feasible commitment policy with respect to the switch times of $\underline{v}(t)$. Since existing algorithms compute a suboptimal $\underline{v}(t)$, that could be used an an initial feasible commitment policy for which the gradient of the cost with respect to the switch times could be used for a gradient search among the switch times for a local minimum around that initial $\underline{v}(t)$. This would result in an improved cost with relatively little added computation. Since current solution techniques discretize the time interval into hour segments, the subsequent gradient search would "fine-tune" the result by allowing for switching the generating units on-line and off-line at any time. It is noted that such a gradient search method is essentially what is used in [3], but for a much simpler model.

The formula for the gradient of $\mathcal{C}[\underline{v}(t)]$ of (15) with respect to the switch times of $\underline{v}(t)$ is given in theorem 8 below. Given an initial feasible commitment function, $\underline{v}_o(t)$, the sets of units which are to be committed between switch times are fixed and only the switch times are varied. If $\underline{v}_o(t)$ has K switch times in (0,T), given by $\{t_k^o\}_{k=1}^K$ with

$$0 < t_1^o < t_2^o < \ldots < t_K^o < T, \tag{149}$$

then $\underline{v}_o(t)$ is given by

$$\underline{v}_o(t) = \underline{v}^k \quad \text{for} \quad t_{k-1}^o \leq t < t_k\ , \ k=1,\ldots,K+1 \tag{150}$$

where $t_o = 0$ and $t_{K+1} = T$. Now, for the fixed sequence of commitment vectors, $\mathcal{V} = \{\underline{v}^k\}_{k=1}^{K+1}$, the switch times determine the scheduling policy and, in turn, the cost. Consider the cost in (15) as a function of the switch times denoted by $\mathcal{C}[\underline{t}^o, \mathcal{V}]$ where

$$\underline{t}^o = [t_1, \ldots, t_K]'. \tag{151}$$

With this minor abuse of notation, the gradient of $\mathcal{C}[\underline{t}, \mathcal{V}]$ is to be found with respect to $\underline{t} \in R^K$. This gradient will be used in the iterative search for the switch times using the iteration formula

$$\underline{t}^{j+1} = \underline{t}^j - \alpha_j \left[\frac{\partial \mathcal{C}}{\partial \underline{t}} \bigg|_{\underline{t}=\underline{t}^j} \right] \tag{152}$$

where

$$\frac{\partial \mathcal{C}}{\partial \underline{t}} \triangleq \left[\frac{\partial \mathcal{C}}{\partial t_1}, \ldots, \frac{\partial \mathcal{C}}{\partial t_K} \right]' \tag{153}$$

and α_j is a scalar chosen at each iteration to satisfy the following two criteria.

(i) The vector $\underline{t}^{j+1}$ defined by (152) must satisfy
$0 < \underline{t}_1^{j+1} < \ldots < t_K^{j+1} < T$.

(ii) $\mathcal{C}(\underline{t}^{j+1}) \le \mathcal{C}(\underline{t}^j)$.

Both (i) and (ii) can always be satisfied by some small enough α [25]. When the improvement in the cost, $\mathcal{C}(t^{j+1}) - \mathcal{C}(\underline{t}^j)$, becomes satisfactorily small, the search is halted. At each iteration a new policy, $\underline{v}_j(t)$, is defined by $\underline{t}^j$ as

$$\underline{v}_j(t) = \underline{v}^k \qquad t_{k-1}^j \le t < t_k^j, \quad k=1,\ldots,K+1 \tag{154}$$

where $t_o^j = 0$ and $t_{K+1}^j = T$.

Finally, to designate the switch times for the individual subsystems, the following definitions are made. For k=1,...,K define the sets Ω_k and Φ_k as

$$\begin{cases} \Omega_k = \{i \mid v_i^k = 1,\ v_i^{k+1} = 0\} \\ \Phi_k = \{i \mid v_i^k = 0,\ v_i^{k+1} = 1\} \end{cases} \tag{155}$$

and for the jth iteration define for i=1,...,N and k=1,...,K,

$$\begin{cases} \lambda_{k,i}^j = \max\{t_\ell^j \mid \ell = 0,\ldots,k-1,\ i \in \Phi_\ell\} \\ \tau_{k,i}^j = \min\{t_\ell^j \mid \ell = k+1,\ldots,K+1,\ i \in \Omega_\ell\} \end{cases} . \tag{156}$$

With these definitions, theorem 8 can be stated giving $\frac{\partial \mathcal{C}}{\partial \underline{t}}$ explicitly.

<u>Theorem 8</u>. For $\mathcal{C}(\underline{t},\mathcal{V})$ defined above, the gradient with respect to $\underline{t} \in R^K$ is given by

$$\frac{\partial \mathcal{C}}{\partial \underline{t}}\Big|_{\underline{t}=\underline{t}^j} = \begin{bmatrix} C[D(t_1^j),\underline{v}_2] - C[D(t_1^j),\underline{v}_1] + \Lambda_1^j \\ \vdots \\ C[D(t_K^j)\underline{v}_{K+1}] - C[D[t_K^j,\underline{v}_K] + \Lambda_K^j \end{bmatrix} \tag{157}$$

where, for k=1,...,K,

$$\Lambda_k^j = \sum_{i \in \Omega_k} \frac{\partial J_i^*}{\partial T}\Big|_{T=t_k^j-\lambda_{k,i}^j} - \sum_{i \in \Phi_k} \frac{\partial J_i^*}{\partial T}\Big|_{T=\tau_{k,i}^j-t_k^j} \tag{158}$$

and $\frac{\partial J_i^*}{\partial T}$ is given for the ith subsystem by (137) of theorem 6.

<u>Proof</u>. Straight forward application of theorems 1 and 6 to the definition (15) of the total cost.

5.2. Modeling of Off-Line Thermal Units

In this section some brief preliminary discussion is presented concerning the dynamic modelling of the thermal generating units while off-line. Previous work on determining start-up costs usually assumes the cost of start-up depends linearly upon the difference between the boiler operating temperature during generation (on-line steady-state operation) and the temperature to which the boiler has cooled since it has been taken off-line. The exponential cooling rate assumed was mentioned above. Since such a model does not include any dynamics which account for the time needed to reheat to boiler to operating temperature, that time factor is usually accounted for by assuming a fixed time, normally an hour, for start-up regardless of the state of the system at the time start-up is initiated.

Since virtually no information concerning dynamic modelling of the start-up process is available in the literature, a simple linear model with a single state is proposed below which takes into account the simpler cost approximations discussed above but also includes a more reasonable and flexible model of the start-up process. Some data from a particular industry application [21] is presented to show the range of values for the cooling time constant, however, data for the start-up time constants are unavailable. Hence, even the simple model suggested below requires further evaluation with respect to current practice.

Consider a lumped parameter model of the boiler dynamics with a single state, τ, representing the average temperature of the water. The temperature has been previously used as the state of the thermal system [12] since in a gross sense it represents the energy content of the system. With no heat input, the cooling rate depends upon the difference between the boiler temperature and the ambient temperature. Following the usual assumption of an exponential cooling rate, if τ_A is the average abient temperature, the differential equation satisfied by τ is

$$\dot{\tau} = -\alpha(\tau - \tau_A)$$

where α is the inverse of the cooling time constant. Reasonable values for α range from 4 to 12 hours [21].

Now, suppose the sole control available is the fuel rate to the burners heating the boiler. If u is the fuel rate in units of volume/time, and β is the conversion constant of heat content per volume of fuel, where it is assumed the heat transfer to the boiler is constant over the operating temperatures, then the single state model, including control is

$$\dot{\tau} = -\alpha(\tau - \tau_A) + \beta u.$$

It is assumed the fuel rate is bounded by

$$0 \leq u \leq \bar{u}.$$

Finally, the operating temperature is denoted by $\bar{\tau}$.

Since it is the difference between the boiler and ambient temperatures which matters, a new state, x, may be defined as

$$x = \tau - \tau_A. \tag{159}$$

Furthermore, let $\bar{x} = \bar{\tau} - \tau_A$. If the cost in \$/volume for the fuel is given by k, then the total cost for operating the generating unit over a time interval [0,T] is given by

$$J(u,T) = \int_{o}^{T} ku\, dt \tag{160}$$

Hence, the optimal control problem for an off-line unit can be stated as follows. Given the off-line interval [0,T], minimize J(u,T) of (157) subject to

$$\dot{x} = -\alpha x + \beta u \tag{161}$$

with the boundary conditions

$$x(0) = \bar{x}, \qquad x(T) = \bar{x}. \tag{162}$$

Clearly the results of Section 4 can be immediately applied to this model. In fact the optimal control $u^*(t,T)$ is given by

$$u^*(t,T) = \begin{cases} 0 & 0 \le t < t_s \\ \bar{u} & t_s \le t \le T \end{cases} \tag{163}$$

where

$$t_s = \frac{1}{\alpha} \lim[1 - \frac{\alpha\bar{x}}{\beta\bar{u}}(1 - e^{-\alpha T})] + T \tag{164}$$

This control produces the optimal trajectory, $x^*(t,T)$, given by

$$x^*(t,T) = \begin{cases} \bar{x}\, e^{-\alpha t} & 0 \le t \le t_s \\ \bar{x}\, e^{-\alpha t} + \frac{\beta\bar{u}}{\alpha}(1 - e^{-\alpha(t-t_s)}) & t_s < t \le T \end{cases} \tag{165}$$

Since the state is a scalar, we have by lemma 16 that the optimal cost as a function of T, $J^*(T)$, is monotone increasing and continuous. From (159) and (163) it is easily shown that $\frac{\partial J^*}{\partial T}$ is given by

$$\frac{\partial J^*}{\partial T} = k\bar{u}[1 + e^{\alpha T}(\frac{\beta\bar{u}}{\alpha\bar{\bar{x}}} - 1)]^{-1} \tag{166}$$

which is always positive.

In words, the optimal policy is to let the boiler cool until the time t_s when there is just a sufficient amount of remaining time before the unit is needed to reheat the boiler to the operating temperature using the maximum fuel rate. Of course, this is roughly what is done in practice. It may be the case, however, that for a more thorough model of the boiler dynamics including time constants for heating boiler walls, multiple burners, circulation rates, and a more detailed lumped parameter model of the governing partial differential equations, the process will not only be better represented but there may be control policies suggested by such an analysis which would improve upon current practice. A detailed modelling of the boiler and turbine dynamics for use in modeling the generating unit while on-line is given in [22] and [23]. This may serve as a general guide for some of the important points which must be considered for a modeling of the start-up process, but there are certain features which are irrelavent to modeling the system while off-line.

It should be noted that one other aspect of the cost of taking units off line and returning them to service which is not included in the above discussion is the fixed costs which may be incurred each time such a switch occurs. Such costs may result from machine wear or labor expenses but these were not included because the added theoretical difficulties which arise in including these factors are far to great compared with the importance and magnitude of such costs. In particular, to include fixed costs for each switch in the continuous model proposed would require the used of δ - functions in the integrand and the theory for optimization with such costs becomes far more complex than that presented above [24]. Because these costs are disregarded it is noted that the optimal solution is never to "bank" the boiler, i.e. to maintain the boiler at its operating temperature while it is off-line. In practice this is, of course, often done and may be the cheapest policy.

5.3. Directions for Further Research

The following list summarizes those areas which were discussed in previous sections as directions for further investigation.

1. Using the results of Section 3 to enhance and improve the guidelines employed by existing algorithms which search among the feasible subsystem combinations for an optimal solution. Any practical algorithms must exclude the majority of options by some method, such as branch and bound techniques when the problem is formulated as an integer programming problem. The exclusion leads to suboptimal solutions and perhaps the knowledge of the properties of the steady state cost function would be useful in determining the appropriate subset of units which should be considered at each demand level.

2. Evaluation of the value of incorporating the suggested gradient search method into existing schemes for optimization. Clearly, with respect to the unit commitment problem, it must be determined whether or not extending existing methods results in enough improvement to justify the added computation. Since all present algorithms lead to suboptimal solutions and in fact may lead to poor solutions under some circumstances, it appears that such an extension is worthy of consideration since it can be incorporated with little change to existing software.

3. Include the probabilistic and stochastic aspects of the problem in the formulation. This would involve essentially defining a new problem. Very basic issues must be resolved concerning appropriate definitions for the cost function under stochastic assumptions as well as defining what is meant by "meeting the demand" or "a feasible set of units". The approach taken for such research should be closely related to the application at hand since even simple formulations quickly lead to complex problems when optimization is attempted and consequently the assumptions made in the model should represent a real problem so that the effort is not meaningless. Nevertheless, it is important that these factors are investigated since it is the case that a strictly deterministic model ignores a fundamental property of the power system application.

4. Include fixed costs for switching units off-line and on-line. As was discussed at the end of Section 5.2, it is questionable that these cost factors merit the added theoretical difficulties incurred by including them, however, research in that direction could be intended to find ways of including these factors in perhaps a more subtle but computationally attractive manner. In practice, since discrete time models are considered, the problems encountered with the continuous time model discussed here do not arise.

5. The practical problem of modeling thermal unit shut-down and start-up. This problem which was discussed at length in Section 5.2 is presently in its infancy and demands an investigation of industry practice. It is a very practical problem which cannot be addressed without close consultation with power system operators.

In summary, this report presents preliminary results and considerations concerning a very complex and important problem. The list given above is merely a sampling of the many appropriate directions for further research addressing the general system coordination problem and the application of unit commitment.

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